Phys 230 2013

Lecture 37

Weds: HW due

Weds: Read:

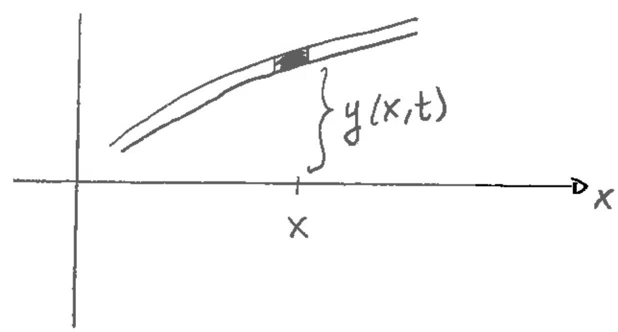
Energy in Waves.

Various waves satisfy the classic wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

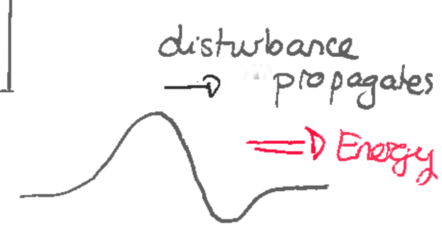
This is rooted in the dynamics of physical scenario. For example, with waves on a string, the term $\frac{\partial^2 y}{\partial t^2}$ is related to the acceleration of a string segment and Newtonian mechanics would describe how this relates to forces acting on the string segment. The ultimate result gives a wavespeed

$$v = \sqrt{T/\mu}.$$



We can also consider energy in waves. When we do this, we shall eventually find a key concept in understanding wave phenomena:

$$\boxed{\text{Waves transport energy in the direction of propagation.}}$$



II. Energy in Waves

1 Energy in Waves on a String

Consider a wave on a string with mass per unit length μ and under tension T. A snapshot of a portion is illustrated in Fig. II.1.1.

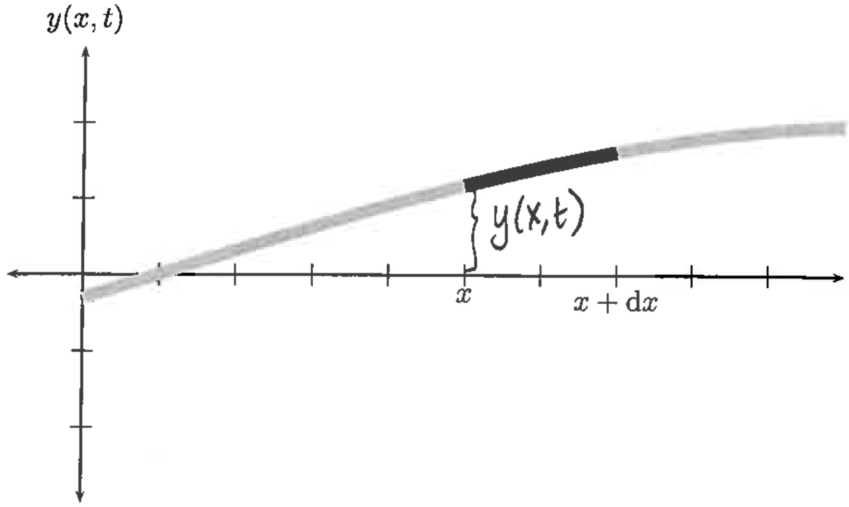


Figure II.1.1: Snapshot of a wave on a string.

The kinetic energy of the illustrated segment depends on the mass of the segment, $\mathrm{d}m$, and its transverse velocity v_y and is

$$\begin{aligned}\mathrm{d}K &= \frac{1}{2}\,\mathrm{d}m v_y^2 \\ &= \frac{1}{2}\,\mathrm{d}m\left(\frac{\partial y}{\partial t}\right)^2\end{aligned} \tag{II.1.1}$$

However, $\mathrm{d}m = \mu \mathrm{d}x$ and thus

$$\mathrm{d}K = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x \tag{II.1.2}$$

Thus the total kinetic energy of a segment of the string from $x = x_1$ to $x = x_2$ is

$$\boxed{\text{Kinetic energy from } x_1 \text{ to } x_2 = K = \frac{1}{2}\int_{x_1}^{x_2} \mu\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x} \tag{II.1.3}$$

We might expect that there is a form of potential energy associated with the deformation of the string. Usually we would obtain this via the work-kinetic energy theorem, $\Delta K = W_{\text{net}}$ where W_{net} is the net work done on a system and ΔK is the change in kinetic energy as the system evolves. It turns out that applying this is more involved than the alternative method that we will offer below. Also, the typical derivation, that yields the correct expression for the potential energy uses a heuristic argument based on the stretch of the string and which has no apparent connection to the work-kinetic energy theorem. We will show that the total potential energy of a segment of the string from $x = x_1$ to $x = x_2$ is

$$\boxed{\text{Potential energy from } x_1 \text{ to } x_2 = U = \frac{1}{2}\int_{x_1}^{x_2} T\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x} \qquad \text{(II.1.4)}$$

where T is the tension in the string. While we will prove that this is a correct expression later, at present we can easily check that this has units of Nm $=$ J, which are the units of energy. With these, the energy in the string from x_1 to x_2 is

$$\boxed{E = \frac{1}{2}\int_{x_1}^{x_2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x.} \qquad \text{(II.1.5)}$$

Related to this we define the energy density (energy per unit length) as

$$\boxed{\text{energy density} = \frac{1}{2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right].} \qquad \text{(II.1.6)}$$

Determining the energy between two point then involves integrating the energy density.

Quiz1

Example: Determine the energy density for the sinusoidal wave

$$y(x,t) = A\cos(kx-\omega t+\phi)$$

Answer: Energy density $= \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$

Note that $v^2 = T/\mu$ and thus $T = v^2\mu$. So

Energy density $= \frac{1}{2}\mu\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]$

Then $\frac{\partial y}{\partial t} = \omega A\sin(kx-\omega t+\phi)$

$\frac{\partial y}{\partial x} = -kA\sin(kx-\omega t+\phi)$

$\Rightarrow$ energy density $= \frac{1}{2}\mu A^2\left[\omega^2 + k^2v^2\right]\sin^2(kx-\omega t+\phi)$

But $kv = \omega$ and so

energy density $= \mu\omega^2A^2\sin^2(kx-\omega t+\phi)$. ☒

Slide 1

Quizz

So the total energy in a sinusoidal wave is infinite. However for pulses of finite spatial extent the energy will be finite.

Slide 2

This indicates that

$\boxed{\text{A wave transports energy. The direction in which energy is transported is the same as the direction of propagation. The velocity with which energy is transported is the same as that of the wave}}$

We can show generally that

If $y(x,t) = f(x-vt)$ then the energy density is:

$$\mu v^2 \left(\frac{df}{du}\right)^2 \bigg|_{u=x-vt}$$

<u>Proof</u>.

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} \qquad u = x - vt$$

$$= -v\frac{df}{du}$$

$$\frac{\partial y}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du}$$

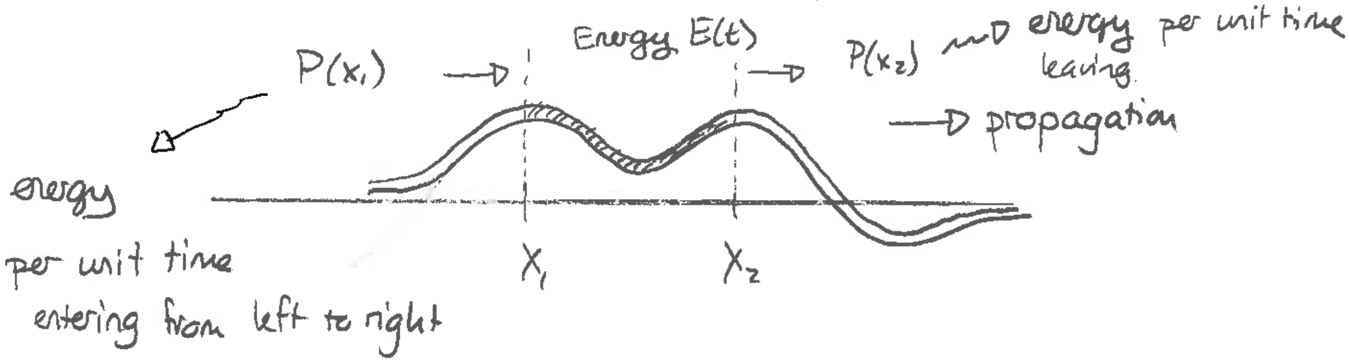
$$\text{energy density} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

$$= \frac{1}{2}\mu v^2\left(\frac{df}{du}\right)^2 + \frac{1}{2}T\left(\frac{df}{du}\right)^2$$

$$= \mu v^2\left(\frac{df}{du}\right)^2$$

and this must be evaluated at $u = x - vt$.

The utility of energy lies in that fact that it may be conserved as time passes. We do not necessarily expect that the energy between x_1 to x_2 will remain constant. Particularly for a pulse of finite width, this will vary as the pulse passes through the region from x_1 to x_2. However, we can establish a relationship between the rate at which energy that enters and leaves the region and the change in total energy in this region.



Specifically,

> Let E be the energy between x_1 to x_2. Then
> $$\frac{dE}{dt} = P(x_1) - P(x_2)$$
> where $P(x)$ is the power transmitted from left to right at x and is given by
> $$P = -\mu v^2 \frac{\partial y}{\partial t}\frac{\partial y}{\partial x}.$$

(II.1.7)

The proof of this starts with Eq. (II.1.5) which gives

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\frac{1}{2}\int_{x_1}^{x_2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x \\
&= \frac{1}{2}\int_{x_1}^{x_2}\left[\mu\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 + T\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x \\
&= \frac{1}{2}\int_{x_1}^{x_2}\left[2\mu\frac{\partial y}{\partial t}\left(\frac{\partial^2 y}{\partial t^2}\right) + 2T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t\partial x}\right]\mathrm{d}x.
\end{aligned} \qquad \text{(II.1.8)}$$

Now $v^2 = T/mu$ gives $T = \mu v^2$ and thus

$$\frac{dE}{dt} = \mu\int_{x_1}^{x_2}\left[\frac{\partial y}{\partial t}\left(\frac{\partial^2 y}{\partial t^2}\right) + v^2\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t\partial x}\right]\mathrm{d}x. \qquad \text{(II.1.9)}$$

Consider the integrand in the second term in this expression

$$\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right) - \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} \qquad \text{(II.1.10)}$$

Thus

$$\begin{aligned}
\frac{dE}{dt} &= \mu \int_{x_1}^{x_2} \left[\frac{\partial y}{\partial t} \left(\frac{\partial^2 y}{\partial t^2} \right) + v^2 \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right) - v^2 \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} \right] \mathrm{d}x \\
&= \mu \int_{x_1}^{x_2} \left\{ \frac{\partial y}{\partial t} \left[\left(\frac{\partial^2 y}{\partial t^2} \right) - v^2 \frac{\partial^2 y}{\partial x^2} \right] + v^2 \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right) \right\} \mathrm{d}x
\end{aligned} \tag{II.1.11}$$

By the wave equation, $\left(\frac{\partial^2 y}{\partial t^2}\right) - v^2 \frac{\partial^2 y}{\partial x^2} = 0$ and thus

$$\begin{aligned}
\frac{dE}{dt} &= \mu v^2 \int_{x_1}^{x_2} \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right) \mathrm{d}x \\
&= \mu v^2 \left(\frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right) \bigg|_{x_2} - \mu v^2 \left(\frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right) \bigg|_{x_1}
\end{aligned} \tag{II.1.12}$$

With the definition of the power transmitted,

$$P = -\mu v^2 \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}$$

we get

$$\frac{dE}{dt} = P(x_1) - P(x_2). \tag{II.1.13}$$

This proves Eq. (II.1.7) and also establishes the validity of the definition of the potential energy, Eq. (II.1.4).

Example: Determine the power transmitted at any point by

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

Answer:

$$P = -\mu v^2 \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$$

$$= -\mu v^2 (-k\omega) A^2 \sin^2(kx - \omega t + \phi)$$

But $\omega = kv \Rightarrow k = \omega/v$ and so

$$P = \mu v \omega^2 A^2 \sin^2(kx - \omega t + \phi)$$ $\blacksquare$

Quiz 3

The power transmitted fluctuates with time. The time-averaged power transmitted is:

$$\bar{P} = \lim_{T \to \infty} \frac{1}{T}\int_0^T P(t)\,dt$$

$$= \;\ldots\; = \frac{1}{2}\mu v \omega^2 A^2$$

We see that this is the same at all points along the wave.