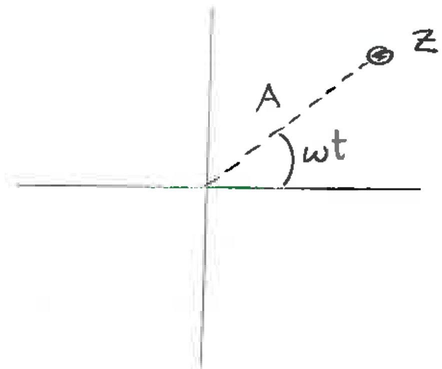
Weds: HW due **Read** 5.1, 5.2

Thurs REU info meeting

Complex numbers and solutions to oscillator equations of motion.

We saw that the complex valued function of time (where A is real)

$$z(t) = Ae^{i\omega t}$$



contains solutions to the equation of motion for a simple harmonic oscillator. This follows from

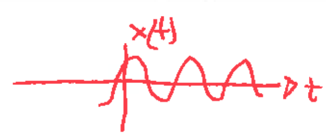
$$z(t) = A\cos(\omega t) + i\,A\sin(\omega t)$$

and so

$$\mathrm{Re}[z(t)] = A\cos(\omega t)$$
$$\mathrm{Im}[z(t)] = A\sin(\omega t)$$

are two distinct solutions to the equation of motion for a simple harmonic oscillator. We see an alternative representation for oscillatory solutions.

Oscillation



complex exponentials	real solutions (actual configuration)
$z(t) = Ae^{i\omega t}$	$x(t) = \mathrm{Re}[z(t)]$

Note that the complex number does not give the configuration (or position). Rather it contains enough information to extract position information.

We need to explore this. We aim to:

1) show that the equation of motion can be solved for a complex solution. This typically eliminates trig identity manipulations.

2) show how to extract physically meaningful quantities directly from the complex solutions. These include:
   * amplitude
   * energy

Before dealing with general solutions + techniques we consider the simple harmonic oscillator.

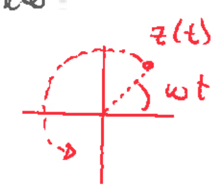
<u>Complex numbers and the simple harmonic oscillator</u>

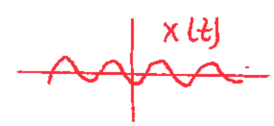
We saw that the general solution to the equation of motion for the simple harmonic oscillator is:

$$x(t) = C\cos(\omega t + \phi)$$

and this is not of the form encountered for $z(t) = Ae^{i\omega t}$ and $x(t) = \mathrm{Re}[z(t)]$. To cast it into this form consider

$$z(t) = \underbrace{C\cos(\omega t+\phi)}_{x(t)\,=\,\mathrm{Re}[z(t)]} + iC\sin(\omega t+\phi)$$





We shall show that this can be expressed as

$$z(t) = De^{i\omega t}$$

for appropriate D.

<u>Quiz 1</u>

Clearly

$$z(t) = C\, e^{i(\omega t + \phi)}$$

$$= C\, e^{i\omega t} e^{i\phi} = C e^{i\phi} e^{i\omega t}$$

Thus with $D = Ce^{i\phi}$ we get $z(t) = De^{i\omega t}$ gives the correct real part. To summarize.

> The solution to the equation of motion for a simple harmonic oscillator can be represented by
>
> $$z(t) = De^{i\omega t}$$
>
> where D is a complex constant. The position or configuration variable can be determined from
>
> $$x(t) = \mathrm{Re}\,[z(t)]$$

We can see that any complex number D works since any complex number D can be written $D = Ce^{i\phi}$ for some real amplitude C and angle ϕ. So our picture becomes

complex representation

$z(t) = De^{i\omega t}$

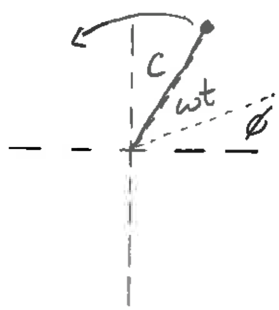
with $D = Ce^{i\phi}$

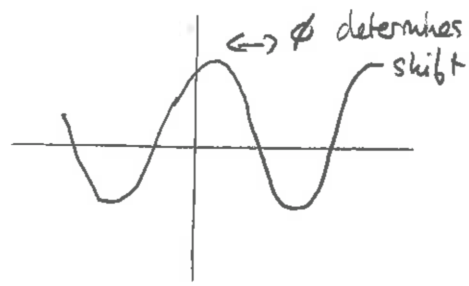
connection

real position / configuration

$x(t) = \mathrm{Re}\,[z(t)]$

$= C\cos(\omega t + \phi)$



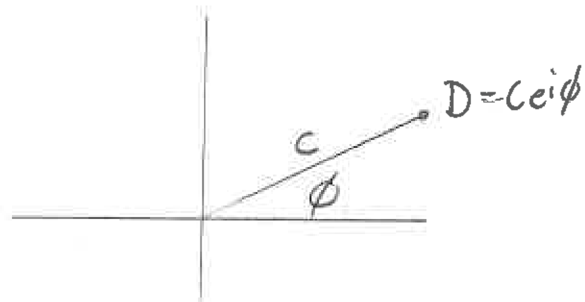


Amplitude, phase and energy.

We see that if

$$z(t) = De^{i\omega t} \quad \text{and} \quad D = Ce^{i\phi}$$

then the amplitude of the oscillation is C and the phase ϕ. We could get these from the complex plane representation of D.

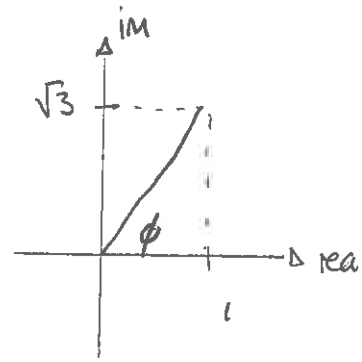


Quiz 2

Example: Determine the phase for the oscillator described by

$$z(t) = (1+\sqrt{3}\,i)\,e^{i12\pi t}$$

Answer: Sketch in the complex plane



Then

$$\tan\phi = \frac{\sqrt{3}}{1}$$

$$\Rightarrow \quad \phi = \arctan\sqrt{3} = \pi/3$$

In general we can see that:

> If $z(t)$ is the complex representation of an oscillation then the amplitude of oscillation is:
>
> $$\text{amplitude} = |z(t)|$$

The phase must be determined from geometrical reasoning.

Quiz 3

The energy of a spring/mass oscillator is:

$$E = \frac{1}{2} M \omega^2 \; C^2$$

and if $z(t) = De^{i\omega t}$ with $D = Ce^{i\phi}$, we have $|z(t)| = C$. Thus we get:

If $z(t)$ is the complex representation of an oscillation then the energy of the oscillator is: $$E = \frac{1}{2} m\omega^2 \, |z(t)|^2$$

This type of relationship appears throughout classical descriptions of the energy of oscillators + waves. It is eventually used as a starting point for describing quantum mechanical versions of oscillators or waves.

Solving equations of motion

The equations of motion for various oscillators all fit the following form:

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

where a_2, a_1, a_0 are constants and $f(t)$ is a function of time. These constants and this function are derived from physical considerations such as Newton's second law. Since we have dealt with complex representations of oscillations, we ask whether there is a complex differential equation that these satisfy. The key is via a theorem.

Theorem: Suppose a_2, a_1, a_0 are real and $z(t)$ is a complex function that satisfies:

$$a_2 \frac{d^2z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z = g(t)$$

Then:

$$x(t) = Re\left[z(t)\right]$$

satisfies:

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

where $f(t) = Re\left[g(t)\right]$

Proof: Note $z(t) = x(t) + iy(t) \Rightarrow \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$

So $Re\left[\frac{dz}{dt}\right] = \frac{dx}{dt} = \frac{d}{dt} Re[z]$ and similarly

$$Re\left[\frac{d^2z}{dt^2}\right] = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} Re[z]$$

Also if a_0 is real $Re\left[a_0 z(t)\right] = Re\left[ax + iay\right] = ax = a\,Re[z]$

Thus $\quad Re\left[a_2 \frac{d^2 z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z\right] = Re[g(t)] = f(t)$

$$\Rightarrow a_2 \frac{d^2}{dt^2} Re[z] + a_1 \frac{d}{dt} Re[z] + a_0 Re[z] = f(t)$$

and this proves the theorem $\boxed{}$

We now have a scheme

<u>Task</u>: Solve $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$ for given $f(t)$

Try various combinations of $\cos(\omega t)$, $\sin(\omega t)$, $e^{\alpha t}$ use trig identities

$\Downarrow$

eventually get $x(t)$

Rewrite a complex function diff eqn:

$$a_2 \frac{d^2z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z = g(t)$$

where $Re[g] = f$

$\Downarrow$

Use $z = De^{ut}$ where u is complex

$\Downarrow$

Algebra determines D, u (no trig).

$\Downarrow$

Real part $x(t) = Re[z(t)]$ satisfies original equation