Phys 230 2013

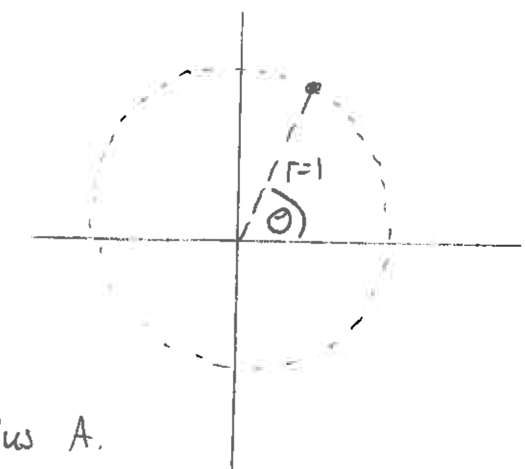
Lecture 33

## Complex numbers, rotations + oscillations

The Euler relation gives:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and this corresponds to a point on a circle of unit radius in the complex plane. We can see that if $\theta = \omega t$ where $\omega$ is real and $t$ represents time that this point rotates along the unit circle. It follows that, when A is real,




$$z(t) = Ae^{i\omega t}$$

corresponds to a point that orbits at radius A.

There is a connection between rotational motion + oscillations

Demo: Rotation / oscillation applet.

The connection is that the horizontal component of a rotating point oscillates left to right and back again. So we should expect a connection between oscillations and complex exponentials. Specifically we will show that:

> If A is real and
>
> $$z(t) = Ae^{i\omega t}$$
>
> then
>
> $$x(t) = \mathrm{Re}[z(t)]$$
> $$y(t) = \mathrm{Im}[z(t)]$$
>
> are both solutions to the equation of motion for a simple harmonic oscillator.

Thus we will see:

> Oscillatory motion is represented by functions of the form $z(t) = Ae^{i\omega t}$ where $A$ is real.

## Complex functions and differential equations.

Complex exponential functions greatly simplify the solution of differential equations that we have encountered. To formalize this we need some generalizations. First we define a general exponential

> If $z$ is any complex number then $z = x+iy$ and
>
> $$e^{z} := e^{x}e^{iy}$$

Then we consider complex valued functions of time;

$$z(t) = x(t) + i\,y(t)$$

and define the derivative of this as:

$$\frac{dz}{dt} := \frac{dx}{dt} + i\,\frac{dy}{dt}.$$

We can then prove that:

> If $D$ is constant and $u$ is any complex number then
>
> $$\frac{d}{dt} De^{ut} = uDe^{ut}$$

<u>Exercise:</u> Starting with $u = x+iy$ and applying the general exponential rule and the rule for differentiation of complex functions, prove this result.

Note that we can use this to solve any differential equation of the form:

$$a_2 \frac{d^2 z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z = 0$$

We try $z = De^{ut}$ Then substitution gives:

$$a_2 u^2 De^{ut} + a_1 Due^{ut} + a_0 De^{ut} = 0$$

$$\Rightarrow \quad a_2 u^2 + a_1 u + a_0 = 0$$

and we solve this for $u$ using

$$u = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

# Intermediate Dynamics: Group Exercises 8

## Complex Numbers and Oscillations

### 1 Complex Functions and Simple Harmonic Motion

a) Let $z(t) = Ae^{i\omega t}$ where $A$ is any real number. Show that the real part of this is a solution to the equation of motion for simple harmonic motion. Repeat this for the imaginary part.

b) Express $z(t) = Ae^{i\omega t}$ in terms of real and imaginary parts and use this to determine an expression for $\frac{dz}{dt}$. Use this and the rules of complex algebra to show that

$$\frac{dz}{dt} = i\omega Ae^{i\omega t} = i\omega z(t).$$

Generalize this to the case where $A$ is replaced by any complex number, $D$.

### 2 Damped Driven Oscillator

The equation of motion for a damped driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos{(\omega t)}.$$

a) Suggest possible complex functions $g(t)$ such that $\text{Re}[g(t)] = \cos{(\omega t)}$. Find a simple function of this type in which trigonometric functions do not appear.

b) Consider

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}. \tag{1}$$

Assume that the solution to this has form

$$z(t) = De^{ut}$$

where $D$ and $u$ are complex constants. Substitute this into Eq. (1) and find algebraic expressions for $u$ and $D$,

c) The complex number $D$ can be represented in the form

$$D = Ae^{-i\delta}$$

where $A$ and $\delta$ are real. Using this, determine an expression for

$$x(t) = \text{Re}[z(t)]$$

and verify that $A$ is the amplitude of oscillation.

d) Show that

$$Ae^{-i\delta} = \frac{F_0}{m} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}.$$

Using the fact that $A = |Ae^{-i\delta}|$ find an expression for $A$ in terms of $F_0, m, \omega, \omega_0, \gamma$.

e) Determine an expression for the phase $\delta$. *(Hint: Use the fact that* $1/z = z^*/|z|^2$*.)*

# Intermediate Dynamics: Group Exercises 8 Solutions
## Complex Numbers and Oscillations

### 1 Complex Functions and Simple Harmonic Motion

a) Let $z(t) = Ae^{i\omega t}$ where $A$ is any real number. Show that the real part of this is a solution to the equation of motion for simple harmonic motion. Repeat this for the imaginary part.

b) Express $z(t) = Ae^{i\omega t}$ in terms of real and imaginary parts and use this to determine an expression for $\frac{dz}{dt}$. Use this and the rules of complex algebra to show that

$$\frac{dz}{dt} = i\omega Ae^{i\omega t} = i\omega z(t).$$

Generalize this to the case where $A$ is replaced by any complex number, $D$.

**Answer:**

*a) Using the Euler relation,*

$$\begin{aligned} z(t) &= Ae^{i\omega t} \\ &= A\left[\cos(\omega t) + i\sin(\omega t)\right] \\ &= A\cos(\omega t) + iA\sin(\omega t) \end{aligned}$$

*Then*

$$Re[z(t)] = A\cos(\omega t)$$

*and this satisfies the equation of motion for simple harmonic motion. Likewise*

$$Im[z(t)] = A\sin(\omega t)$$

*also satisfies the equation of motion.*

*b) Again*

$$z(t) = A\cos(\omega t) + iA\sin(\omega t).$$

*Thus*

$$\begin{aligned} \frac{dz}{dt} &= \frac{d}{dt}\left[A\cos(\omega t) + iA\sin(\omega t)\right] \\ &= A\left[-\omega\ \sin(\omega t) + i\omega\cos(\omega t)\right] \\ &= i\omega A\left[i\sin(\omega t) + \cos(\omega t)\right] \\ &= i\omega Ae^{i\omega t} = i\omega z(t). \end{aligned}$$

Exercise 2

a) $g(t) = e^{i\omega t}$ has $Re[g(t)] = \cos(\omega t)$

b) $$\frac{dz}{dt} = u D e^{ut}$$

$$\frac{d^2 z}{dt^2} = u^2 D e^{ut}$$

$$\Rightarrow \quad u^2 D e^{ut} + u\gamma D e^{ut} + \omega_0^2 D e^{ut} = \frac{F_0}{m} e^{i\omega t}$$

$$\Rightarrow \quad (u^2 + u\gamma + \omega_0^2) D e^{ut} = \frac{F_0}{m} e^{i\omega t}$$

So we need $u = i\omega$ — (1)

$$(u^2 + u\gamma + \omega_0^2) D = \frac{F_0}{m} \qquad -(2)$$

Substitute (1) into (2):

$$(-\omega^2 + i\omega\gamma + \omega_0^2) D = F_0/m$$

c) $$Z(t) = Ae^{-i\delta} e^{i\omega t}$$

$$= Ae^{i(\omega t - \delta)} = A\cos(\omega t - \delta) + iA\sin(\omega t - \delta)$$

$$Re[z(t)] = A\cos(\omega t - \delta)$$

amplitude

d) $(-\omega^2 + i\omega\gamma + \omega_0^2) Ae^{-i\delta} = F_0/M$

$(\omega_0^2 - \omega^2 + i\omega\gamma) Ae^{-i\delta} = F_0/M$

$$Ae^{-i\delta} = \frac{F_0}{M} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

$$A = |Ae^{-i\delta}| = \frac{F_0}{M} \frac{1}{|\omega_0^2 - \omega^2 + i\omega\gamma|}$$

$$= \frac{F_0}{M} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

e) $$Ae^{-i\delta} = \frac{F_0}{M} \frac{\omega_0^2 - \omega^2 - i\omega\gamma}{[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]}$$ using $\frac{1}{z} = \frac{z^*}{|z|^2}$

$$\Rightarrow \cancel{\frac{F_0}{M}} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} e^{-i\delta} = \cancel{\frac{F_0}{M}} \frac{\omega_0^2 - \omega^2 - i\omega\gamma}{[\ \ldots\ldots\ ]}$$

$$\Rightarrow e^{-i\delta} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2) + \omega^2\gamma^2}} \left[(\omega_0^2 - \omega^2) + i\omega\gamma\right].$$

$\Downarrow$

$$\cos\delta - i\sin\delta = \ldots\ldots$$

$$\Rightarrow \frac{\sin\delta}{\cos\delta} = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \Rightarrow \tan\delta = \frac{\ldots}{\ldots}$$

$$\Rightarrow \delta = \arctan\left[\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right]$$