Friday 3.6

Monday New HW!

Library references

Complex numbers

The mathematics of trigonometric functions, that appear in oscillations, turns out to be closely related to complex numbers.

Complex numbers are an abstract generalization of real numbers. The key new ingredient is an imaginary number, the square root of -1. We use the symbol

$$i = \sqrt{-1}$$

and the real meaning of this resides in multiplication, specifically.

$$\boxed{i^2 = -1.}$$

A complex number is a pair of real numbers (x, y) that are usually written:

$$\boxed{z = \underbrace{x}_{\substack{\text{real part of } z \\ x = Re[z]}} + i\underbrace{y}_{\substack{\text{imaginary part of } z \\ y = Im[z].}}}$$

Note that every real number is also a complex number (with the imaginary part $= 0$).

This provides a definition of the set of all complex numbers. What we now require are definitions of algebraic operations on these numbers.

These operations involve:

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

and are:

1) <u>equality</u> $\quad z_1 = z_2 \iff x_1 = x_2$ <u>AND</u> $y_1 = y_2$

2) <u>addition</u> $\quad z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

3) <u>multiplication</u> $\quad z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

We can see that multiplication is defined so that distribution holds:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\ &= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + i(x_1 y_2 + x_2 y_1) - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)\end{aligned}$$

<u>Quiz1</u> $4/7 \to 7/7$

There are additional useful operations associated with complex numbers:

<u>Complex conjugation:</u> If $z = x + iy$ then the complex conjuate is

$$z^* = x - iy$$

<u>Quiz2</u> $5/7$

Note that

1) $\boxed{z \text{ is real} \iff z^* = z}$

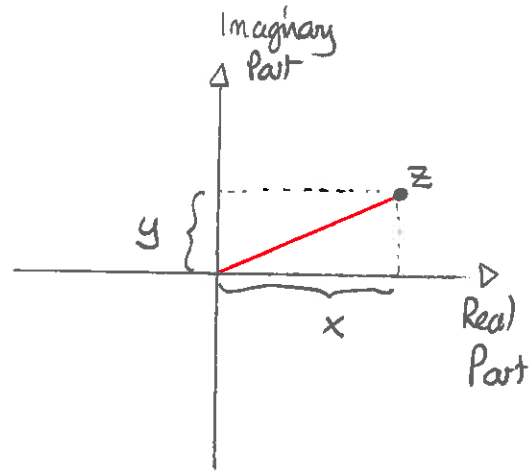
2) For any complex numbers z_1, z_2

$$\boxed{(z_1 z_2)^* = z_1^* z_2^*}$$

Exercise: prove this

<u>Complex plane:</u>

We can represent complex numbers as points in a two dimensional plane since they are pairs of real numbers. The real and imaginary parts are oriented as illustrated. Then the number $z = x+iy$ is represented by the indicated point.



This gives rise to the notion of modulus, which is the distance from the origin to z. Thus we define

$$\boxed{\text{The modulus of } z = x+iy \text{ is } |z| = \sqrt{x^2+y^2}}$$

We can show that

1) $|z|^2 = z^* z$

2) $|z^*| = |z|$

3) $|z_1 z_2| = |z_1||z_2|$

Quiz 3 5/7

Quiz 4

Complex Inversion

The inverse of z, denoted z^{-1} or $1/z$, must satisfy

$$z z^{-1} = 1$$

Now if $|z| \neq 0$ then we can see:

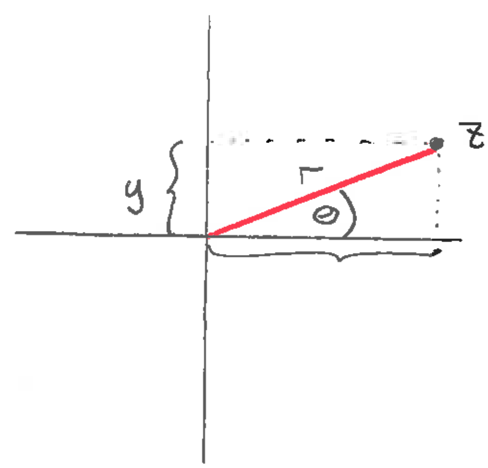
$$\boxed{\frac{1}{z} = z^{-1} = \frac{z^*}{|z|^2}}$$

since $$z\left(\frac{z^*}{|z|^2}\right) = \frac{z z^*}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

Polar representation of complex numbers

Since complex numbers can be represented in the complex plane with Cartesian co-ordinates, they must have a polar co-ordinate representation. We use the standard polar variables r, θ.

We can see



$$z = x + iy$$

$$\Rightarrow \quad x = r\cos\theta$$
$$y = r\sin\theta$$

so

$$z = r(\cos\theta + i\sin\theta)$$

Here it is clear that

$$r = |z|$$

We now introduce the complex exponential notation:

$$\boxed{\begin{array}{l}\text{Given any real number } \theta, \\ e^{i\theta} := \cos\theta + i\sin\theta\end{array}}$$

Euler relation.

Quiz 5 6 7 8 9 10

mostly 90%

Thus

$$\boxed{z = r e^{i\theta}}$$

The Euler relation stands as a definition of a complex exponential. However, assorted trigonometric identities can be used to show that it obeys many of the usual rules of exponentiation. These include:

$$\boxed{\begin{array}{lll} e^{i\theta} e^{i\phi} & = e^{i(\theta+\phi)} & \text{for any real } \theta, \phi \\ e^{i0} & = 1 & \\ 1/e^{i\theta} & = e^{-i\theta} & \end{array}}$$

Aside from this complex conjugation can be applied:

$$\boxed{(e^{i\theta})^* = e^{-i\theta}}$$

Proof: $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow (e^{i\theta})^* = (\cos\theta + i\sin\theta)^* = \cos\theta - i\sin\theta$

But $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

So $e^{-i\theta} = (e^{i\theta})^*$ ∎

Also

$$\boxed{|e^{i\theta}| = 1}$$

as this lies on a circle of radius 1.

<u>Proof</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

$$|e^{i\theta}| = \cos^2\theta + \sin^2\theta = 1 \quad \text{for any } \theta.$$

The complex exponential admits the same Taylor series expansion as the usual exponential. So

$$\boxed{e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots = \sum_n (i\theta)^n/n!}$$

<u>Proof</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

$$= \left[1 - \theta^2/2! + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \theta^8/8! \dots\dots\right] \leftarrow \text{expansion for } \cos\theta$$

$$+ i\left[\theta - \theta^3/3! + \theta^5/5! \dots\right] \leftarrow \text{expansion for } \sin\theta$$

$$= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} \dots\dots \quad \blacksquare$$

Lastly there are methods of expressing trigonometric functions in terms of complex exponentials

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

These are useful for proving trigonometric identities.

Example: Use the complex exponential to prove the sin and cos addition formula:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

Answer: $e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$

But $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$

$$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi$$

$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

The real parts must be equal. So

$$\cos\theta\cos\phi - \sin\theta\sin\phi = \cos(\theta+\phi)$$

The imaginary parts must also be equal. So

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \sin(\theta+\phi) \quad \boxed{}$$