Phys 230 2013

Lecture 31

Weds: 3.6 Space Science Course

Driven damped oscillator

The equation of motion for a driven spring/mass system is

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \qquad -(1)$$

Here ω_0 is the natural frequency of the oscillator $\left(\omega_0 = \sqrt{\frac{k}{m}}\right)$. This is completely determined by system parameters. The other frequency present, ω is the frequency of the driving force and this can be varied in a controllable way. The solution to the equation of motion is:

$$x(t) = A(\omega)\cos(\omega t - \delta)$$

where the amplitude of oscillation is:

$$A(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} \qquad -(1)$$

and the phase is:

$$\delta = \arctan\left[\frac{\omega\gamma}{\omega_0^2-\omega^2}\right]$$

The phase describes the offset between the displacement and the driving force.

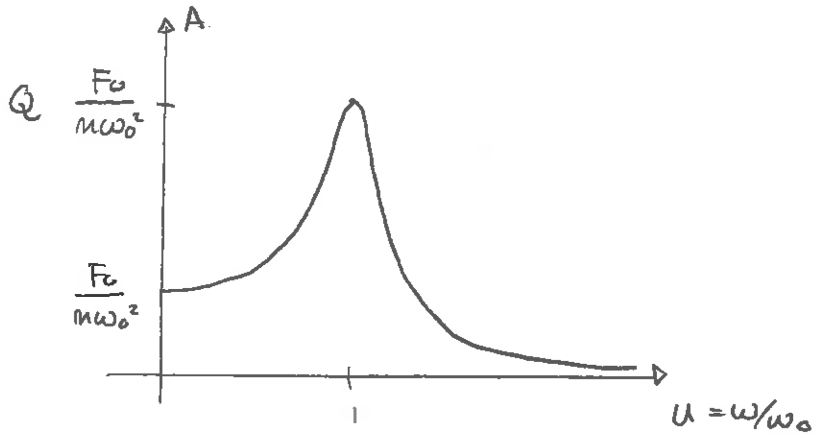
Both of these can be rephrased in terms of a dimensionless parameter that describes ω. Let

$$\boxed{u := \omega/\omega_0}$$

Then:

$$\boxed{\begin{aligned} A &= \frac{F_0}{m\omega_0^2}\,\frac{1}{u\sqrt{(u-1/u)^2+1/Q^2}} \\ \delta &= \arctan\left[\frac{1}{Q}\,\frac{u}{1-u^2}\right]\end{aligned}}$$

where $Q = \omega_0/\gamma$. This facilitates plotting. We find that the amplitude displays a peak when $u \approx 1$ (i.e. $\omega \approx \omega_0$). The extent of the peak depends on Q.



Demo: PhET Resonance
* Show one resonator, various frequs
  - 0.1 Hz
  - 0.75 Hz
  - 2 Hz.

This peaking is called resonance.

Note phase

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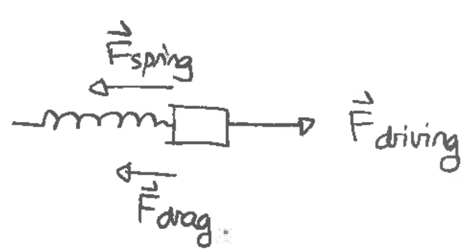
Force + power delivered.

The driving force supplies energy to the system. The energy absorbed will depend on the duration for which the driving force acts. Thus we will consider the rate at which it delivers energy.

First we must distinguish between various forces acting on the mass. The net force is:

$$\vec{F}_{net} = \vec{F}_{drag} + \vec{F}_{spring} + \vec{F}_{driving}.$$
$$\parallel$$
$$m\vec{a}$$



We can access information about this via

$$F_{net\,x} = m a_x \quad \Rightarrow \quad F_{net} = m \frac{d^2x}{dt^2}$$

What concerns us here is the energy delivered by the driving force. We need a lemma:

> The instantaneous power provided by a force $\vec{F}$ acting on an object is:
>
> $$P = \vec{F}(t) \cdot \vec{v}(t)$$
>
> where $\vec{v}(t)$ is the velocity of the object.

Proof: The work done by a force acting on an object from time $t_0 \to t$ is:

$$W = \int_{t_0}^{t} \vec{F} \cdot d\vec{r}$$

But $$\frac{d\vec{r}}{dt} = \vec{v} \quad \Rightarrow \quad d\vec{r} = \vec{v} \cdot dt.$$

So

$$W = \int_{t_0}^{t} (\vec{F} \cdot \vec{v}) dt'$$

Then the power delivered is:

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \qquad \blacksquare$$

For the damped driven oscillator that moves in one dimension, the power delivered by the driving force is:

$$P = F_{driving} \times v_x$$

$$= F_0 \cos(\omega t)\, v(t)$$

But $v(t) = \frac{dx}{dt} = -\omega A(\omega) \sin(\omega t - \delta)$. Thus the instantaneous power delivered is

$$P = -\omega F_0 A(\omega) \cos(\omega t) \sin(\omega t - \delta)$$

Thus

> The instantaneous power delivered by the external driving force is:
>
> $$P(t) = -\frac{F_0^2}{m\omega_0} \frac{1}{\sqrt{(u - 1/u)^2 + 1/Q^2}} \cos(\omega t) \sin(\omega t - \delta)$$
>
> where $u = \omega/\omega_0$

This has the following features:

1) $P(t)$ fluctuates with time
2) Sometimes P is positive, at other times P is negative
3) the extent of fluctuation depends on the phase δ

To understand the fluctuations, recall:

$$\omega \ll \omega_0 \Rightarrow \delta = 0$$

$$\omega = \omega_0 \Rightarrow \delta = \pi/2$$

$$\omega \gg \omega_0 \Rightarrow \delta = \pi$$

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We can see that

$$\delta = 0 \Rightarrow \cos(\omega t)\sin(\omega t - \delta) = \cos(\omega t)\sin(\omega t) = \tfrac{1}{2}\sin(2\omega t)$$

$$\delta = \pi \Rightarrow \cos(\omega t)\sin(\omega t - \delta) = \cos(\omega t)\sin(\omega t - \pi)$$
$$= -\cos(\omega t)\sin(\omega t) = -\tfrac{1}{2}\sin(2\omega t)$$

$$\delta = \pi/2 \Rightarrow \cos(\omega t)\sin(\omega t - \delta) = \cos(\omega t)\sin(\omega t - \pi/2)$$
$$= \cos(\omega t)\left[\sin(\omega t)\cos\pi/2 - \cos(\omega t)\sin(\pi/2)\right]$$
$$= -\cos^2(\omega t)$$

Clearly the most efficient average delivery is when $\delta = \pi/2$ (i.e. at resonance).

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We capture this fluctuating power in terms of an average power delivered over a long time. This is defined as

> Let $T > 0$. Then the time averaged power is
>
> $$\bar{P} = \lim_{T \to \infty} \frac{1}{T} \int_0^T P(t)\,dt$$

Then we can show that

> For a damped driven oscillator
>
> $$\bar{P} = \frac{F_0^2}{2m\omega_0} \frac{1}{Q} \frac{1}{\left[(1/u - u)^2 + \frac{1}{Q^2}\right]}$$
>
> where $u = \omega/\omega_0$, $Q = \omega_0/\gamma$.

Proof: Supplied later.

We can also show that, for these sinusoidally driven oscillations,

$$\bar{P} = \frac{1}{T}\int_{t_0}^{t_0+T} P(t)\,dt$$

where T is the period of oscillation.

Quiz 4

We can see that

1) As $u \to 0$ $(\omega = 0)$ $P \to 0$

$u \to \infty$ $(\omega \to \infty)$ $P \to \infty$

2) $\boxed{\text{The maximum power is attained at resonance } (\omega = \omega_0)}$

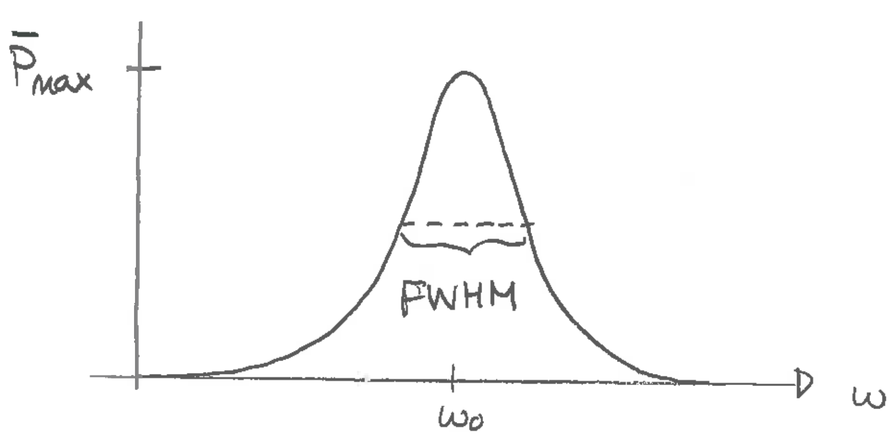
and

$$\bar{P}_{max} = \frac{F_0^2}{2m\omega_0} Q$$

3) The power is:

$$\bar{P} = \frac{\bar{P}_{max}}{Q^2} \frac{1}{\left[(1/u - u)^2 + \frac{1}{Q^2}\right]}$$

4) A plot gives a near symmetrical curve



The width of the curve is quantified by finding

1) values of ω s.t. $\bar{P} = \bar{P}_{max}/2$

2) determining the range of such values.

This is called the full-width half-max. (FWHM).

Then

$$\boxed{FWHM = \gamma}$$

<u>Proof:</u> $\bar{P} = \bar{P}_{max}/2$ when u satisfies:

$$\frac{\bar{P}_{max}}{2} = \frac{\bar{P}_{max}}{Q^2} \frac{1}{\left[(1/u - u)^2 + 1/Q^2\right]}$$

$$\Rightarrow \quad 2/Q^2 = \left(\frac{1}{u} - u\right)^2 + 1/Q^2 \quad \Rightarrow \quad \frac{1}{Q^2} = \left(\frac{1}{u} - u\right)^2$$

$$\Rightarrow \quad \pm \frac{1}{Q} = \left(\frac{1}{u} - u\right)$$

$$\Rightarrow \quad 1 - u^2 = \pm\, u/Q \quad \Rightarrow \quad u^2 \pm u/Q - 1 = 0$$

For + sign $\quad u_+ = \dfrac{-1/Q \pm \sqrt{1/Q^2 + 4}}{2} = \dfrac{-1/Q + \sqrt{1/Q^2 + 4}}{2}$

For − sign $\quad u_- = \dfrac{+1/Q \pm \sqrt{1/Q^2 + 4}}{2} = \dfrac{1/Q + \sqrt{1/Q^2 + 4}}{2}$

$$u_- - u_+ = \frac{1}{Q} \quad \Rightarrow \quad \frac{\omega_-}{\omega_0} - \frac{\omega_+}{\omega_0} = \frac{1}{Q} = \gamma/\omega_0$$

$$\Rightarrow \quad \Delta\omega = \gamma \quad \blacksquare$$

Proof. (of time averaged power)

$$\bar{P} = \lim_{T\to\infty} \frac{1}{T}\int_0^T P(t)\,dt$$

Now $$P(t) = \frac{-F_0^2}{m\omega_0} \frac{1}{\left[(u-1/u)^2 + 1/Q^2\right]^{1/2}} \cos(\omega t)\sin(\omega t - \delta)$$

and $$\delta = \arctan\left[\frac{u}{Q(1-u^2)}\right]$$

Then $P(t) = B\cos(\omega t)\left[\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta\right]$ where $B = \frac{-F_0^2}{m\omega_0}\frac{1}{[\,\ldots\,]^{1/2}}$. So we need to integrate

$$\frac{1}{T}\int_0^T \cos(\omega t)\sin(\omega t)\,dt = \frac{1}{T}\int_0^T \frac{\sin(2\omega t)}{2}\,dt = \frac{-1}{2T\,2\omega}\cos(2\omega t)\Big|_0^T$$

$$\to 0 \text{ as } T\to\infty$$

Also

$$\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \frac{1}{T}\int_0^T \frac{\cos(2\omega t)+1}{2}\,dt$$

$$= \frac{1}{2T}\int_0^T \left[1+\cos(2\omega t)\right]dt$$

$$= \frac{1}{2T}\left[t + \frac{1}{2\omega}\sin(2\omega t)\right]_0^T$$

$$= \frac{1}{2T}\left[T + \frac{1}{2\omega}\sin(2\omega T)\right] = \frac{1}{2} + \frac{1}{4\omega T}\sin(2\omega T)$$

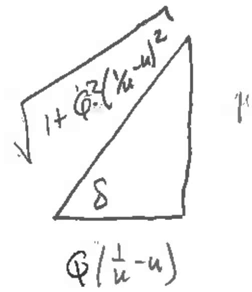
$$\to 1/2 \text{ as } T\to\infty$$

Thus

$$\bar{P} = -B\sin\delta\,\frac{1}{2}$$

We need an expression for $\sin\delta$. Note that

$$\tan\delta = \frac{u}{Q(1-u^2)} = \frac{1}{Q(1/u - u)}$$



implies a triangle of the indicated type. Thus

$$\sin\delta = \frac{1}{\sqrt{1+Q^2(1/u-u)^2}} = \frac{1}{Q}\,\frac{1}{\sqrt{(1/u-u)^2 + 1/Q^2}}$$

So

$$\bar{P} = \frac{F_0^2}{2m\omega_0}\,\frac{1}{Q}\,\frac{1}{\left[(u-1/u)^2 + 1/Q^2\right]} \qquad \blacksquare$$

A slight alternative is:

$$\tan\delta = \frac{\sin\delta}{\cos\delta} \Rightarrow \tan^2\delta = \frac{\sin^2\delta}{\cos^2\delta} \Rightarrow \tan^2\delta\cos^2\delta = \sin^2\delta$$

$$\Rightarrow \tan^2\delta(1-\sin^2\delta) = \sin^2\delta$$

$$\Rightarrow \sin^2\delta(1+\tan^2\delta) = \tan^2\delta$$

$$\Rightarrow \sin^2\delta = \frac{\tan^2\delta}{1+\tan^2\delta} = \frac{1}{1+1/\tan^2\delta}$$

So $$\sin^2\delta = \frac{1}{1+Q^2(1/u-u)^2} \Rightarrow \sin\delta = \frac{1}{\sqrt{1+Q^2(1/u-u)^2}}$$

where the positive root is taken since $0 \le \delta \le \pi$