Phys 230 2013

Lecture 30

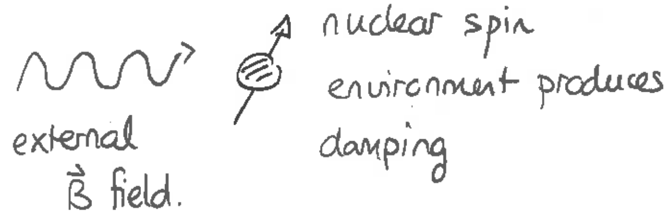
Mon: HW due

Mon: Read 3.3, 3.4, 3.6

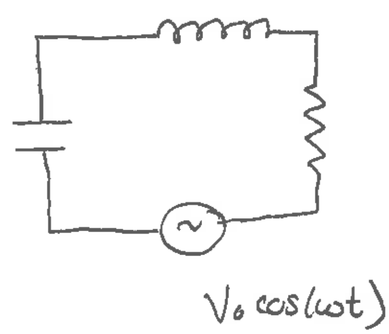
Driven damped oscillator

In many situations a damped oscillator is driven by a controllable external force. For example.

1) magnetic resonance



2) driven RLC circuit



The mathematics of these is identical to that of a spring mass system that is driven. The equation of motion is:

$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)} \qquad -(1)$$

where ω_0 = natural frequency

ω = controllable driving frequency.

Again, we would like to:

1) determine steady state solutions to the equation of motion.

2) determine amplitudes of oscillation and how these depend on driving frequency

3) determine energy absorbed by the oscillator

We attempt a trial solution and find:

The steady-state solution to (1) is:

$$x(t) = A(\omega)\cos(\omega t - \delta)$$

where the amplitude of oscillation is:

$$A(\omega) = \frac{F_0/M}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}$$

and the phase is:

$$\delta = \arctan\left[\frac{\omega\gamma}{\omega_0^2-\omega^2}\right]$$

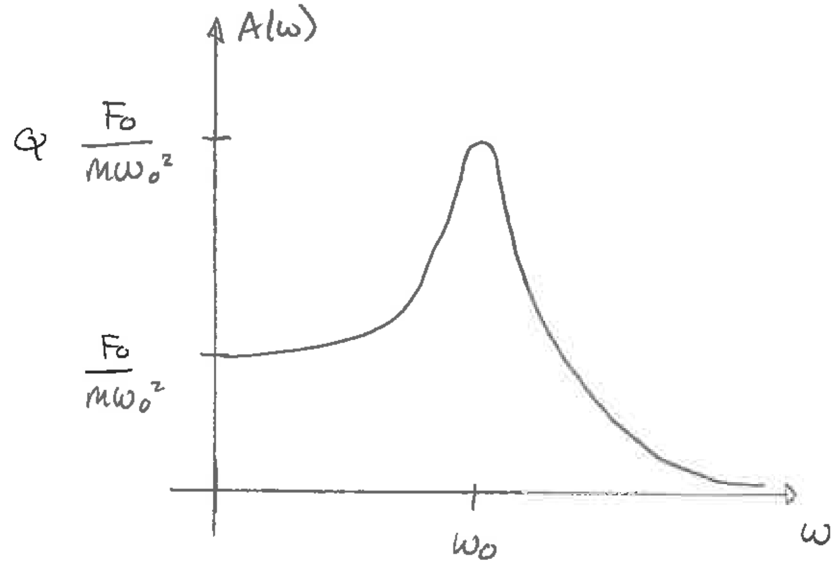
The amplitude varies with driving frequency.

1) as $\omega \to 0$ $\quad A(\omega) \to F_0/M\omega_0^2$

2) as $\omega \to \infty$ $\quad A(\omega) \to 0$

3) as $\omega \to \omega_0$ $\quad A(\omega) \to \dfrac{F_0}{M\omega_0^2\,\gamma/\omega_0} = \dfrac{F_0}{M\omega_0^2}\,Q$

We find a peak near $\omega = \omega_0$



This peaked amplitude behavior when $\omega \approx \omega_0$ is called resonance.

Intermediate Dynamics: Group Exercises 7

Driven Damped Oscillator

1 Solutions to the equation of motion

The equation of motion for a damped driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \tag{1}$$

Consider the following candidate for the steady state solution to the equation of motion:

$$x(t) = A\cos(\omega t - \delta).$$

a) A preliminary step involves rewriting the right hand side of Eq. (1). Note that $\cos(\omega t) = \cos(\omega t - \delta + \delta)$. Use this and a trigonometric identity to rewrite $\cos(\omega t)$ as a linear combination of $\cos(\omega t - \delta)$ and $\sin(\omega t - \delta)$.

b) Substitute $x(t)$ into the rewritten version of Eq. (1) and express the result as a linear combination of $\cos(\omega t - \delta)$ and $\sin(\omega t - \delta)$. Use this to determine conditions that A and δ must satisfy. Solve these to get expressions for A and δ.

2 Phase of oscillation for a driven damped oscillator.

The phase of a driven damped oscillator is.

$$\delta = \arctan\left[\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right].$$

The aim of this exercise is to explore this phase as a function of driving frequency and relate it to the motion of the oscillator.

a) Rewrite δ in terms of Q and $u = \omega/\omega_0$.

b) Suppose that $\omega \to 0$. Determine an approximate expression for δ. What does your result imply for the motion of the oscillator relative to the driving force?

c) Suppose that $\omega \to \infty$. Determine an approximate expression for δ. What does your result imply for the motion of the oscillator relative to the driving force?

d) Suppose that $\omega = \omega_0$. Determine an approximate expression for δ. What does your result imply for the motion of the oscillator relative to the driving force?

e) Sketch a plot of δ as a function of u.

Ex 7

Question 1

a) $$\cos(\omega t) = \cos\left[(\omega t-\delta)+\delta\right]$$

$$= \cos(\omega t-\delta)\cos\delta - \sin(\omega t-\delta)\sin\delta$$

b) $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t-\delta)\cos\delta - \frac{F_0}{m}\sin(\omega t-\delta)\sin\delta$$

$$\frac{dx}{dt} = -\omega A\sin(\omega t-\delta)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t-\delta)$$

$$\Rightarrow \quad -\omega^2 A\cos(\omega t-\delta) - \gamma\omega A\sin(\omega t-\delta) + \omega_0^2 A\cos(\omega t-\delta)$$

$$= \frac{F_0}{m}\cos\delta\cos(\omega t-\delta) - \frac{F_0}{m}\sin\delta\sin(\omega t-\delta)$$

$$\Rightarrow \left[(\omega_0^2-\omega^2)A - \frac{F_0}{m}\cos\delta\right]\cos(\omega t-\delta)$$

$$+ \left[-\gamma\omega A + \frac{F_0}{m}\sin\delta\right]\sin(\omega t-\delta) = 0$$

This is only true at all times if:

$$(\omega_0^2-\omega^2)A = \frac{F_0}{m}\cos\delta$$

$$\omega\gamma A = \frac{F_0}{m}\sin\delta$$

Immediately we get:

$$\frac{\cancel{\frac{F_0}{m}} \sin\delta}{\cancel{\frac{F_0}{m}} \cos\delta} = \frac{\omega\gamma}{\omega_0^2-\omega^2} \Rightarrow \tan\delta = \frac{\omega\gamma}{\omega_0^2-\omega^2}$$

$$\Rightarrow \delta = \arctan\left[\frac{\omega\gamma}{\omega_0^2-\omega^2}\right]$$

Now

$$\left(\frac{F_0}{m}\cos\delta\right)^2 + \left(\frac{F_0}{m}\sin\delta\right)^2 = (\omega_0^2-\omega^2)^2 A^2 + \omega^2\gamma^2 A^2$$

$$\Rightarrow \left(\frac{F_0}{m}\right)^2 \underbrace{\left[\cos^2\delta + \sin^2\delta\right]}_{=1} = \left[(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2\right] A^2$$

$$\Rightarrow \quad A = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

2 Phase of oscillation for a driven damped oscillator.

The phase of a driven damped oscillator is.

$$\delta = \arctan\left[\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right].$$

The aim of this exercise is to explore this phase as a function of driving frequency and relate it to the motion of the oscillator.

a) Rewrite δ in terms of Q and $u = \omega/\omega_0$.

b) Suppose that $\omega \to 0$. Determine an approximate expression for δ. What does your result imply for the motion of the oscillator relative to the driving force?

c) Suppose that $\omega \to \infty$. Determine an approximate expression for δ. What does your result imply for the motion of the oscillator relative to the driving force?

d) Suppose that $\omega = \omega_0$. Determine an approximate expression for δ. What does your result imply for the motion of the oscillator relative to the driving force?

e) Sketch a plot of δ as a function of u.

a) Note that $Q = \omega_0/\gamma$ gives $\gamma = \omega_0/Q$. Thus

$$\begin{aligned}
\delta &= \arctan\left[\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right] \\
&= \arctan\left[\frac{\omega\omega_0/Q}{\omega_0^2 - \omega^2}\right] \\
&= \arctan\left[\frac{\omega\omega_0}{\omega_0^2 Q}\frac{1}{1 - \omega^2/\omega_0^2}\right] \\
&= \arctan\left[\frac{\omega}{\omega_0 Q}\frac{1}{1 - \omega^2/\omega_0^2}\right] \\
&= \arctan\left[\frac{u}{Q}\frac{1}{1 - u^2}\right] \\
&= \arctan\left[\frac{1}{Q}\frac{u}{1 - u^2}\right].
\end{aligned}$$

b) If $\omega \to 0$ then $u \to 0$. Thus

$$\delta \to \arctan\left[\frac{u}{Q}\right] \to \arctan 0 = 0.$$

The oscillator moves in step with the driving force.

c) If $\omega \to \infty$ then $u \to \infty$. Thus

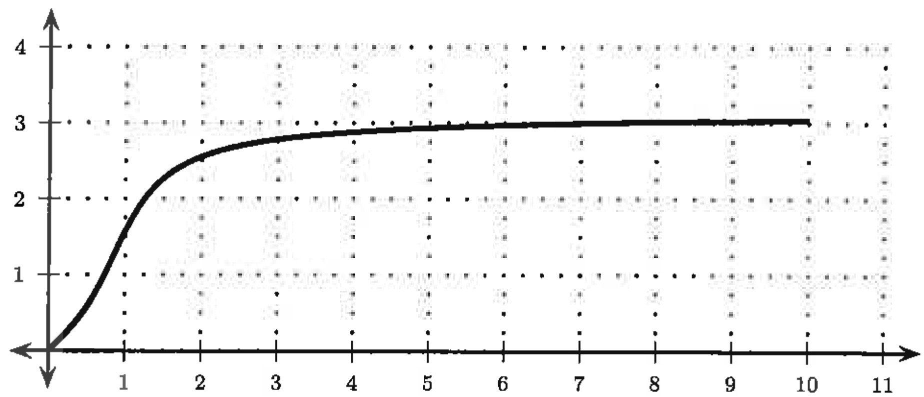
$$\delta \to \arctan\left[\frac{u}{-Qu^2}\right] = \arctan\left[\frac{1}{-Qu}\right] \to \pi$$

since the argument of arctan is negative. The oscillator opposite to the driving force.

d) Here $u = 1$ *and*

$$\delta \to \arctan[\infty] = \frac{\pi}{2}.$$

e) This depends on the value of Q*. For* $Q = 1$,



For $Q = 100$,