Weds : Turn in HW

Fri: Read 3.2.2

Mon HW due.

Damped oscillator : RLC circuit.

The equation of motion for a damped oscillator is:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

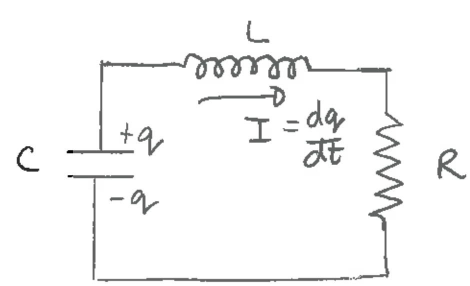
where, for a spring/mass system: $\gamma = b/m$, $\omega_0^2 = k/m$. The energy of this system is

$$E = \alpha\left\{\left(\frac{dx}{dt}\right)^2 + \omega_0^2 x^2\right\}$$

where for a spring/mass system $\alpha = \frac{1}{2}m$. Given this the rate at which energy is dissipated is

$$\frac{dE}{dt} = -2\alpha\gamma\left(\frac{dx}{dt}\right)^2 \qquad \left(= -b\left(\frac{dx}{dt}\right)^2 \text{ for spring/mass system}\right)$$

One example of an oscillating system with damping is an RLC series circuit. The inductor + capacitor allow for oscillations and the resistor provides damping. The standard analysis for this involves expressing voltage drops around the circuit in terms of charge. An alternative involves energy. Let E be the energy stored in the inductor and capacitor.



So $E = E_L + E_C$

$$= \frac{1}{2} L I^2 + \frac{1}{2}\frac{1}{C} q^2$$

$$= \frac{1}{2} L \left\{ \left(\frac{dq}{dt}\right)^2 + \frac{1}{LC} q^2 \right\}$$

Then the resistor dissipates this. The power dissipated is $P = VI$ and so

$$\frac{dE}{dt} = -P = -VI = -I^2R = -R\left(\frac{dq}{dt}\right)^2$$

We see that the energy term and the energy dissipation term have the same form as that for the generic oscillator where $\alpha = \frac{1}{2}L$, $\omega_0^2 = \frac{1}{LC}$

$-2\alpha\gamma = -R \Rightarrow \gamma = \frac{R}{2\alpha} = \frac{R}{L}$. Thus the charge satisfies:

$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC} q = 0}$$

We see

$$\boxed{\gamma = R/L \qquad \omega_0^2 = \frac{1}{LC}.}$$

Alternatively:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C} q = 0$$

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For crictical damping $\omega_0 = \gamma/2 \Rightarrow \frac{1}{\sqrt{LC}} = \frac{R}{2L} \Rightarrow R = 2\sqrt{\frac{L}{C}}$

The oscillations will be lightly damped if $R < 2\sqrt{L/C}$.

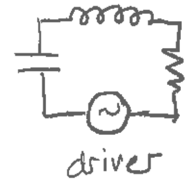
Forced oscillations

We have so far considered circumstances where an oscillator, which is isolated, is set into motion at an initial instant and is then subsequently allowed to oscillate in isolation. But many oscillators are also constantly forced by an outside system.

Demo: suspended spring/mass

Other examples include

a) driven electrical circuits



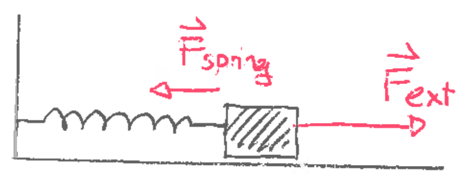
b) driven oscillating strings / columns of air in musical systems.

c) charges in atomic systems driven by electromagnetic waves.

d) magnetic resonance.

What we aim for when considering these is how the system responds to the external driving force in terms of:

1) frequency of oscillation
2) amplitude of oscillation
3) energy delivered / absorbed.



We start by considering a spring/mass system. Newton's 2nd Law gives

$$\vec{F}_{net} = m\vec{a} \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = F_{spring\,x} + F_{ext\,x}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -kx + F_{ext\,x}$$

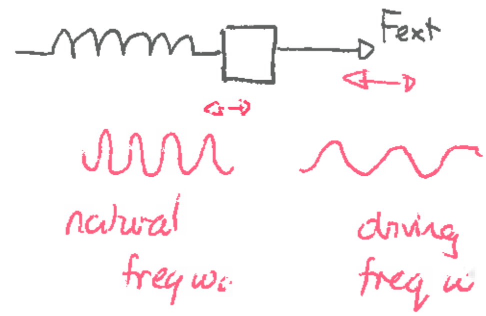
$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\omega_0^2 x + \frac{F_{ext\,x}}{m} \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Sinusoidal driving force

The solution to the equation of motion depends on the nature of the driving force. One very typical type of driving force is a sinusoidally varying force whose frequency is controlled from the outside. Thus we consider

$$F_{ext} = F_0 \cos(\omega t)$$

amplitude of force — driving frequency



The equation of motion is:

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)} \qquad -(1)$$

The general solution can be expressed as:

$$x(t) = x_t(t) + x_s(t)$$

"transient" solution
- ~ short term behavior
- ~ contains constants determined by initial conditions
- ~ $x_t(t) = C\cos(\omega_0 t + \phi)$

"steady state" solution
- ~ long-term behavior
- ~ contains no free constants

We shall focus on the steady state solutions. Also note that these will generally only make sense when damping is present.

Steady-state solution

We shall attempt a solution of the form

$$x_s(t) \equiv x(t) = A\cos(\omega t - \delta) \quad -(2)$$

where $A > 0$ is a constant and δ is a constant.

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Exercise: Check that (2) satisfies (1)

Answer:
$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t - \delta)$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2 x = [\omega_0^2 - \omega^2] A\cos(\omega t - \delta)$$

But this satisfies (1) if it equals $\frac{F_0}{m}\cos(\omega t)$. So we get

$$[\omega_0^2 - \omega^2] A\cos(\omega t - \delta) = \frac{F_0}{m}\cos(\omega t)$$

$$\Rightarrow A\cos(\omega t - \delta) = \frac{F_0}{m}\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t).$$

So if $\omega_0 > \omega$ then (2) satisfies (1) if

$$A = \frac{F_0}{m}\frac{1}{\omega_0^2 - \omega^2} \quad \text{and } \delta = 0$$

and if $\omega_0 < \omega$ then (2) satisfies (1) if

$$A = \frac{F_0}{m}\frac{1}{\omega^2 - \omega_0^2} \quad \text{and} \quad \delta = \pi$$

Using $\cos(\omega t - \pi) = -\cos(\omega t)$ gives:

The equation of motion for a driven oscillator

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{M}\cos(\omega t)$$

has a steady state solution,

$$x(t) = \frac{F_0}{M}\frac{1}{\omega_0^2-\omega^2}\cos(\omega t)$$

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We see:

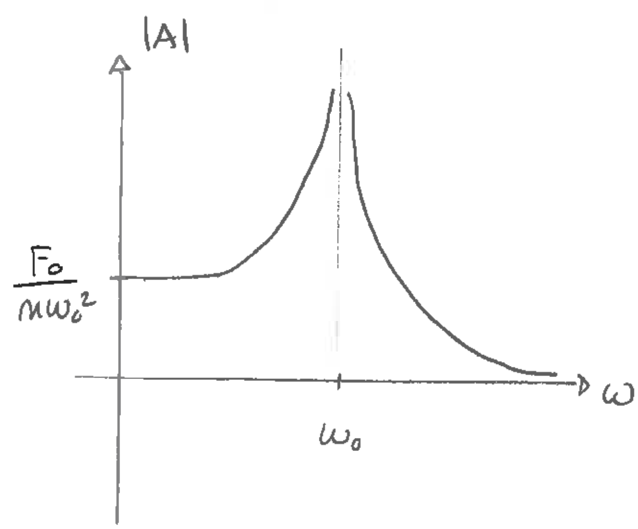
$$\boxed{\text{Amplitude of oscillation} = |A| = \frac{F_0}{M}\frac{1}{|\omega_0^2-\omega^2|}}$$

This depends on the driving force and the driving frequency in relation to the natural frequency. We see:

1) as $\omega \to 0$ $\quad |A| \to \frac{F_0}{M\omega_0^2}$

2) as $\omega \to \infty$ $\quad |A| \to 0$

3) as $\omega \to \omega_0$ $\quad |A| \to \infty$



Demo: PhET resonance