Phys 230 2013

Lecture 28

HW due Monday
HW due Weds
Weds: 2.4, 3.1, 3.2.1

Damped Oscillations

The equation of motion for a damped oscillator is:

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 \qquad (1)$$

where γ is the damping constant and ω_0 is the natural frequency. We have already found one type of solution:

If $\gamma < 2\omega_0$ then the general solution to (1) is:

$$x(t) = Ce^{-\gamma t/2}\cos(\omega t + \phi)$$

where

$$\omega = \sqrt{\omega_0^2 - \gamma^2/4}$$

is the frequency of oscillation and C, ϕ are constants

This is a lightly damped oscillation.

Slide 1

However, there must be solutions for $\gamma \geq 2\omega_0$. The text describes a process by which trial solutions are guessed. These are substituted into Eq. (1) and result algebra fixes various parameters.

We find:

> If $\gamma > 2\omega_0$ then the general solution is:
>
> $$x(t) = e^{-\gamma t/2}\left[Ae^{\alpha t} + Be^{-\alpha t}\right]$$
>
> where A, B are constants and
>
> $$\alpha = \sqrt{\gamma^2/4 - \omega_0^2}$$

This gives a decaying solution which $\to 0$ as $t \to \infty$ (since both exponentials $e^{-(\gamma/2-\alpha)t}$, $e^{-(\gamma/2+\alpha)t}$ are negative. This is called a <u>heavily damped solution</u>.

Slide 2

Lastly

> If $\gamma = 2\omega_0$ then
>
> $$x(t) = (A+Bt)\,e^{-\gamma t/2}$$

where A, B are constants. This is called critically damped.

Slide 3

<u>Properties of critically damped solutions.</u>

1) <u>As $t \to \infty$ $x \to 0$</u> We need L'Hopital's rule

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{f'(t)}{g'(t)}$$

where $f(t) = A + Bt$ $\Rightarrow$ $f'(t) = B$

$g(t) = e^{\gamma t/2}$ $\quad g'(t) = \gamma/2\, e^{\gamma t/2}$

So $\lim \frac{f'}{g'} = 0$

2) <u>Rate of damping vs heavily damped</u>

Consider two oscillators with same damping constants but different natural frequencies. The critically damped is :

$$x_c(t) = (A + Bt)e^{-\gamma t/2}$$

The heavily damped is

$$x_h(t) = e^{-\gamma t/2}\left(\tilde{A}e^{\alpha t} + \tilde{B}e^{-\alpha t}\right)$$

Assume that both are at rest at $t=0$ and have the same velocity $V_0 = V(0)$. Then $x(0) = 0$ for each. So.

$$x_c(0) = 0 \Rightarrow A = 0 \qquad \Rightarrow x_c = Bte^{-\gamma t/2}$$

and

$$\frac{dx_c}{dt} = Be^{-\gamma t/2} + Bt\,e^{-\gamma t/2}\left(-\gamma/2\right)$$

$$\Rightarrow \quad V_0 = B$$

So $\qquad x_c(0) = V_0 t e^{-\gamma t/2}$

For the heavily damped, $x_h(0) = 0 \Rightarrow \tilde{A} = -\tilde{B}$

$$\Rightarrow \quad x_h(t) = e^{-\gamma t/2}\tilde{A}\left[e^{\alpha t} - e^{-\alpha t}\right].$$

Now

$$v(t) = \frac{dx_h}{dt} = -\gamma/2\, e^{-\gamma t/2}\tilde{A}\left[e^{\alpha t} - e^{-\alpha t}\right] + \tilde{A}e^{-\gamma t/2}\alpha\left[e^{\alpha t} + e^{-\alpha t}\right]$$

$$\Rightarrow V_0 = \tilde{A}\alpha 2 \Rightarrow \quad \tilde{A} = \frac{V_0}{2\alpha}$$

So

$$x_h(t) = \frac{V_0}{2\alpha}e^{-\gamma t/2}\left[e^{\alpha t} - e^{-\alpha t}\right].$$

Thus

$$\frac{x_c(t)}{x_h(t)} = 2\alpha \frac{1}{e^{\alpha t} - e^{-\alpha t}}$$

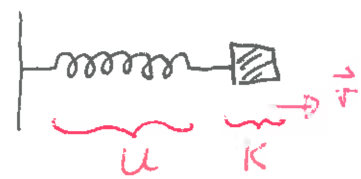
and we can see that as $t \to \infty$ $\quad \frac{x_c(t)}{x_h(t)} \to 2\alpha e^{-\alpha t}$

So $x_c(t)$ approaches zero faster than $x_h(t)$

Slide 4

Energy and damped oscillations

Consider a spring and mass system with damping. The mechanical energy includes contributions from:



1) the spring – potential energy $U = \frac{1}{2}kx^2$
2) the mass – kinetic energy $K = \frac{1}{2}mv^2$

The total mechanical energy is

$$E = K + U = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

or

$$\boxed{E = \frac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2 + \omega_0^2 x^2\right\}} \qquad -(2)$$

Now, the drag force always opposes motion and so it constantly does negative work on the block. So the drag force constantly reduces the mechanical energy. We aim to assess this for a lightly damped oscillator. Specifically we would like:

1) general expressions for the energy at any time
2) expressions for the rate at which energy decays or dissipates
3) a notion of how well oscillations persist (determined via energy considerations).

The basic results can be obtained by direct substitution

With the solution to the equation of motion for lightly damped oscillations

$$x(t) = Ce^{-\gamma t/2}\cos(\omega t + \phi)$$

the mechanical energy is:

$$E(t) = e^{-\gamma t}E_0\left\{1 + \frac{1}{\omega_0^2}\left[\frac{\gamma^2}{4}\cos(2\omega t + 2\phi) + \frac{\gamma\omega}{2}\sin(2\omega t + 2\phi)\right]\right\} \qquad -(3)$$

where $E_0 = \frac{1}{2}m\omega_0^2 C^2$

Exercise: Show this

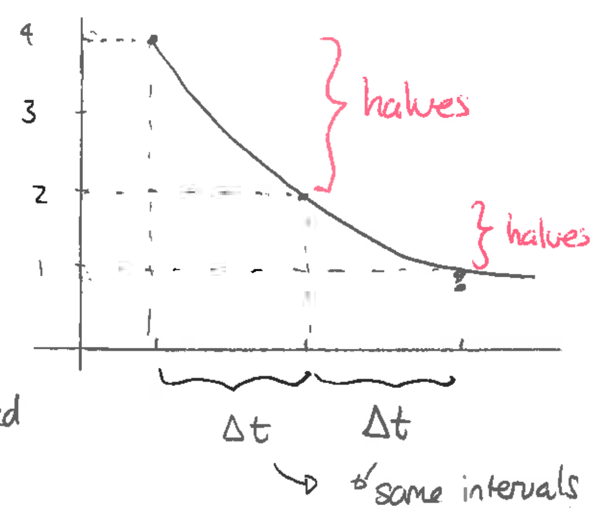
Here E_0 is the energy of the undamped oscillator. We can see that the energy contains oscillating terms but it also contains an exponential decay term with decay constant γ (rather than $\gamma/2$).

For very light damping the oscillating terms are irrelevant. Specifically suppose $\omega_0 >> \gamma/2$. Then $\omega \approx \omega_0$ and we get that the factors in (3) $\frac{\gamma^2}{4\omega_0^2} \ll 1$, $\frac{\gamma\omega}{2\omega_0^2} \ll 1$. So

$$\boxed{\begin{array}{l} \text{If } \omega_0 >> \gamma/2 \text{ then} \\ \quad E(t) = E_0 e^{-\gamma t} \\ \text{where } \quad E_0 = \frac{1}{2} m\omega^2 C^2 \text{ and} \\ \quad E(t) = \frac{1}{2} m\omega^2 C(t)^2 \\ \text{where } \quad C(t) = C e^{-\gamma t/2} \end{array}} \quad -(4)$$

Plotting would yield an exponential decay curve for energy with a decay constant γ (double the decay constant for amplitude). Note that for any initial time t_0 and time interval Δt

$$\frac{E(t_0+\Delta t)}{E(t_0)} = e^{-\gamma \Delta t}$$



and the fractional decay does not depend on the initial instant but rather on the elapsed time and the decay rate. Note

$$\ln\left[\frac{E(t_0+\Delta t)}{E(t_0)}\right] = -\gamma \Delta t \Rightarrow \quad \gamma = \frac{1}{\Delta t} \ln\left[\frac{E(t_0)}{E(t_0+\Delta t)}\right]$$

Quiz 1 $\quad 1/5$

t	E
1s	16 J
3s	8 J
5s	4 J
7s	2 J

We see that in a succession of equally spaced intervals the energy repeatedly halves. An example is given.

Then the decay constant quantifies this rate of energy decay.

Quiz 2 $\quad 1/5 \to 5/5$

Exact energy decay rate.

We can combine the generic equation of motion for a damped oscillator

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

with the mechanical energy

$$E = \alpha \left\{ \left(\frac{dx}{dt}\right)^2 + \omega_0^2 x^2 \right\}$$

α = constant that depends on system.

to yield

$$\frac{dE}{dt} = \alpha \left\{ \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 + \omega_0^2 \frac{d}{dt}(x^2) \right\}$$

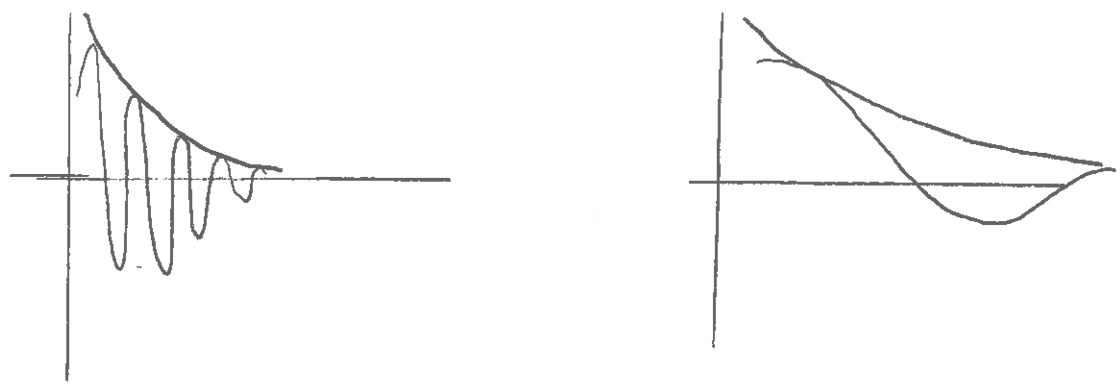
$$= \alpha \left\{ 2\frac{d^2x}{dt^2}\frac{dx}{dt} + \omega_0^2 2 \frac{dx}{dt} x \right\} = 2\alpha \underbrace{\left\{ \frac{d^2x}{dt^2} + \omega_0^2 x \right\}}_{= -\gamma \frac{dx}{dt}} \frac{dx}{dt}$$

$$\boxed{\frac{dE}{dt} = -2\alpha\gamma\left(\frac{dx}{dt}\right)^2}$$

This is clearly ≤ 0 and so energy will generally be lost. This rule applies to any solution for the damped oscillator.

Quality factor.

The decay rate captures information about the sheer decay in energy. But it does not describe the persistence of oscillations well.



decay more rapid
oscillations more rapid
persist better

decay less rapid
oscillations less rapid

We can address this in terms of energy decay per cycle. Suppose T is the period of oscillation (i.e. $\omega T = 2\pi$). Then

$$\frac{E(t_0+T)}{E(t_0)} = e^{-\gamma T} = e^{-2\pi\gamma/\omega}$$

For light damping $\omega \approx \omega_0$ and so

$$\frac{E(t_0+T)}{E(t_0)} = e^{-2\pi\gamma/\omega_0}$$

We define the quality factor

$$Q := \omega_0/\gamma$$

→ oscillation rate

→ energy decay rate.

and so

$$\boxed{\frac{E(t_0+T)}{E(t_0)} = e^{-2\pi/Q}} \quad \text{and} \quad \frac{C(t_0+T)}{C(t_0)} = e^{-\pi/Q}$$

Quiz