Mon: HW due

Weds: Read Ch 2.

Damped Oscillations

The solution to the equation of motion for a simple harmonic oscillator is

$$x(t) = C\cos(\omega t + \phi)$$

This describes oscillations that persist indefinitely. The energy of this oscillator is given by

$$E = \alpha\left[\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2\right]$$

where α is a constant. Substitution from the general solution yields.

$$E = \alpha C^2\omega^2$$

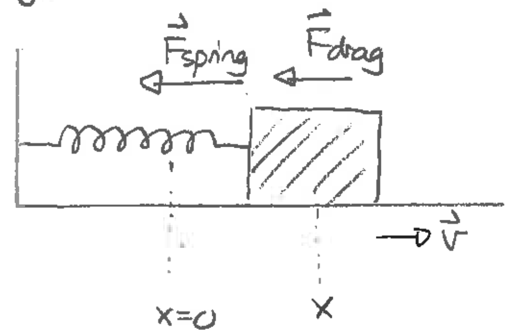
Exercise: Prove the relationship above. ☒

The energy of the oscillator stays constant.

In general observations suggest that oscillations do not persist indefinitely and that energy dissipates. So the equation of motion for a simple harmonic oscillator does not reflect this dissipation. We need to modify it accordingly, so as to yield oscillations that are damped.

Equation of motion for damped oscillations.

Again consider a spring + mass system but suppose that there is an additional drag force proportional to velocity.



We assume that this has horizontal component

$$F_{drag} = -bv = -b\frac{dx}{dt}$$

where $b>0$ is a constant.

Newton's 2nd Law gives:

$$\vec{F}_{net} = m\vec{a}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

and this gives the equation of motion for a damped oscillators:

$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0}$$

where $\gamma = b/m$ is the damping constant.

$\omega_0 = \sqrt{k/m}$ is the natural frequency.

This second order differential equation is again linear.

Solving the equation of motion

We seek oscillating damped solutions. These have the form

$$x(t) = Ce^{-\alpha t}\cos(\omega t + \phi)$$

where $\omega > 0$ is not necessarily ω_0 and $\alpha > 0$ is a constant.

By direct substitution we can show:

1) such solutions only exist when $\gamma < 2\omega_0$

2) $\alpha = \gamma/2$

3) $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$

So we get

> If $\gamma < 2\omega_0$ then the general solution to the equation of motion
>
> $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$
>
> is
>
> $$x(t) = C e^{-\gamma t/2} \cos(\omega t + \phi)$$
>
> where
>
> $$\omega = \sqrt{\omega_0^2 - \gamma^2/4}$$

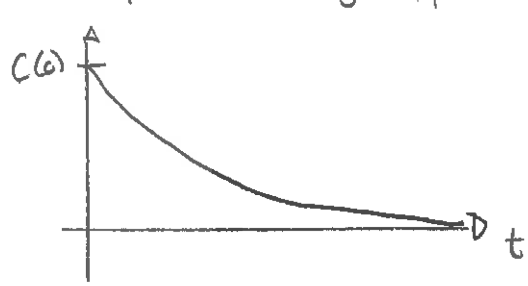
This is called lightly damped or underdamped oscillations. We can rewrite this with a time dependent amplitude.

$$\boxed{x(t) = C(t)\cos(\omega t + \phi)}$$

where

$$\boxed{C(t) = Ce^{-\gamma t/2}}$$

The amplitude decays exponentially.



Properties of this solution.

1) Period: The angular frequency ω differs from the natural frequency and $\omega < \omega_0$.

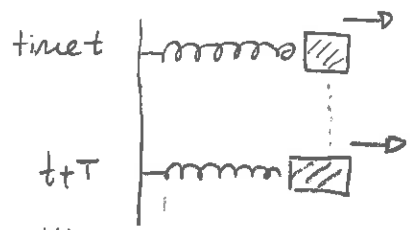
The previous definition of period, i.e. the shortest time T s.t $x(t+T) = x(t)$ for all t has no meaning here since the function does not repeat. But we could define the period as

$$\boxed{\text{The period is the shortest time between successive moments when the oscillator is at equilibrium and is moving in the same direction.}}$$

These occur when

$x(t) = 0 \Rightarrow \cos(\omega t + \phi) = 0$

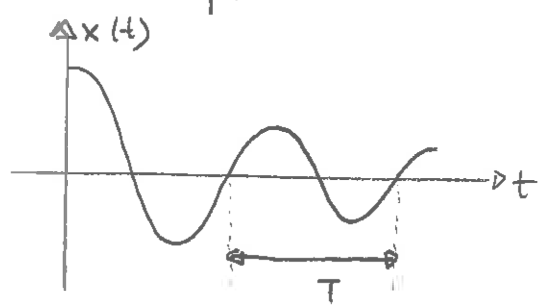
and $\dot{x}(t)$ has same sign. We see that this means



$$\omega(t+T) + \phi = \omega t + \phi + 2\pi$$

$$\Rightarrow \omega T = 2\pi$$

$$\Rightarrow \quad T = \frac{2\pi}{\omega}$$



2) Maxima: Maxima occur when

$$\frac{dx}{dt} = 0 \Leftrightarrow C\left(-\gamma/2\right)e^{-\gamma t/2}\cos(\omega t + \phi) - C\omega e^{-\gamma t/2}\sin(\omega t + \phi) = 0$$

$$\Leftrightarrow \quad \gamma/2 \cos(\omega t + \phi) + \omega \sin(\omega t + \phi) = 0$$

If one maximum occurs at time t', then at $t'+T$

$$\gamma/2 \cos(\omega(t'+T)+\phi) + \omega \sin(\omega(t'+T)+\phi)$$

$$= \gamma/2 \cos(\omega t' + 2\pi + \phi) + \omega \sin(\omega t' + \phi + 2\pi)$$

$$= \gamma/2 \cos(\omega t' + \phi) + \omega \sin(\omega t' + \phi)$$

But if a max occurs at t' then this expression is zero. So

$$\gamma/2 \cos(\omega(t'+T)+\phi) + \omega \sin(\omega(t'+T)+\phi) = 0$$

which is also a max. Thus

$\boxed{\text{Successive maxima are separated by intervals equal to the period.}}$

Intermediate Dynamics: Group Exercises 6

Damped Harmonic Oscillator

1 Damped harmonic oscillator

The equation of motion for a damped harmonic oscillator is

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0. \tag{1}$$

We shall aim to find damped oscillations described by

$$x(t) = Ce^{-\alpha t} \cos(\omega t)$$

where $\alpha > 0$ and $\omega > 0$ are constants *(note that the symbols ω and ω_0 are intentionally different).*

a) Assuming this type of solution determine an expression for $\frac{dx}{dt}$.

b) Determine an expression for $\frac{d^2x}{dt^2}$.

c) Substitute these into Eq. (1) and gather all terms with $\cos(\omega t)$ and all with $\sin(\omega t)$. Determine conditions on α and ω such that the resulting equation is satisfied at all times.

d) Is this solution valid for all values of γ and ω_0?

Exercise 1

a) $\frac{dx}{dt} = C(-\alpha)e^{-\alpha t}\cos(\omega t) + Ce^{-\alpha t}(-\omega)\sin \omega t$

$= -Ce^{-\alpha t}\left[\alpha \cos(\omega t) + \omega \sin(\omega t)\right]$

b) $\frac{d^2x}{dt^2} = -C(-\alpha)e^{-\alpha t}\left[\alpha \cos(\omega t) + \omega \sin(\omega t)\right]$

$-Ce^{-\alpha t}\left[-\alpha\omega \sin(\omega t) + \omega^2 \cos(\omega t)\right]$

$= Ce^{-\alpha t}\left\{\cos(\omega t)\left[\alpha^2-\omega^2\right] + \sin(\omega t)\, 2\alpha\omega\right\}$

c)

$Ce^{-\alpha t}\left\{\left[\alpha^2-\omega^2\right]\cos(\omega t) + 2\alpha\omega \sin(\omega t)\right\}$

$+ \gamma(-C)e^{-\alpha t}\left[\alpha \cos(\omega t) + \omega \sin(\omega t)\right]$

$+ \omega_0^2 Ce^{-\alpha t}\cos(\omega t) \qquad = 0$

$\Rightarrow Ce^{-\alpha t}\left\{\cos(\omega t)\left[\alpha^2-\omega^2-\gamma\alpha+\omega_0^2\right] + \sin(\omega t)\left[2\alpha\omega - \gamma\omega\right]\right\} = 0$

$\Rightarrow \cos(\omega t)\left[\alpha^2-\omega^2-\gamma\alpha+\omega_0^2\right] + \omega(2\alpha-\gamma)\sin(\omega t) = 0.$

At $t=0$ we get $\alpha^2-\omega^2-\gamma\alpha+\omega_0^2=0$ and this is true for all times.
So $\omega(2\alpha-\gamma)\sin(\omega t)=0$. This requires $\omega(2\alpha-\gamma)=0$. So

$2\alpha-\gamma=0 \Rightarrow \boxed{\alpha=\gamma/2}$

$\alpha^2-\omega^2-\gamma\alpha+\omega_0^2=0 \Rightarrow \gamma^2/4 - \omega^2 - \gamma^2/2 + \omega_0^2 = 0 \Rightarrow \omega^2 = \omega_0^2 - \gamma^2/4$

$\Rightarrow \boxed{\omega = \sqrt{\omega_0^2-\gamma^2/4}}$

d) No we need $\omega_0^2 \geq \gamma^2/4 \Rightarrow \boxed{\omega_0 > \gamma/2}$

2 Lightly damped oscillations

The amplitude for a lightly damped oscillator is

$$C(t) = Ce^{-\gamma t/2}$$

where C is a constant.

a) Determine $C(1\,\mathrm{s})/C(0\,\mathrm{s})$.

b) Determine $C(2\,\mathrm{s})/C(1\,\mathrm{s})$.

c) Determine $C(t+1\,\mathrm{s})/C(t)$ for any time t.

d) Determine γ from the ratio $C(t+1\,\mathrm{s})/C(t)$ for any time t.

e) Determine γ from the ratio $C(t+T)/C(t)$ where T for any time t but where T is the period of oscillation.

Answer:

a) From the definition

$$\frac{C(1\,s)}{C(0\,s)} = \frac{C_0 e^{-\gamma 1/2}}{C_0 e^{-\gamma 0/2}} = e^{-\gamma/2}.$$

b) From the definition

$$\frac{C(2\,s)}{C(1\,s)} = \frac{C_0 e^{-\gamma 2/2}}{C_0 e^{-\gamma 1/2}} = \frac{e^{-\gamma}}{e^{-\gamma/2}} = e^{-\gamma+\gamma/2} = e^{-\gamma/2}.$$

c) From the definition

$$\frac{C(t+1\,s)}{C(t)} = \frac{C_0 e^{-\gamma(t+1)/2}}{C_0 e^{-\gamma t/2}} = \frac{e^{-\gamma t/2-\gamma/2}}{e^{-\gamma t/2}} = e^{-\gamma t/2-\gamma/2+\gamma t/2} = e^{-\gamma/2}.$$

Notice that the fractional reduction in amplitude over any $1\,s$ *interval is the same regardless of when the interval started. Then*

$$\ln\left(\frac{C(t+1\,s)}{C(t)}\right) = \ln\left(e^{-\gamma/2}\right)$$

gives

$$\ln\left(\frac{C(t+1\,s)}{C(t)}\right) = -\frac{\gamma}{2}$$

and thus

$$\gamma = -2\,\ln\left(\frac{C(t+1\,s)}{C(t)}\right) = 2\,\ln\left(\frac{C(t)}{C(t+1\,s)}\right)$$

d) A similar derivation gives

$$\frac{C(t+T)}{C(t)} = \frac{C_0 e^{-\gamma(t+T)/2}}{C_0 e^{-\gamma t/2}} = e^{-\gamma T/2}.$$

Again, using natural logarithms,

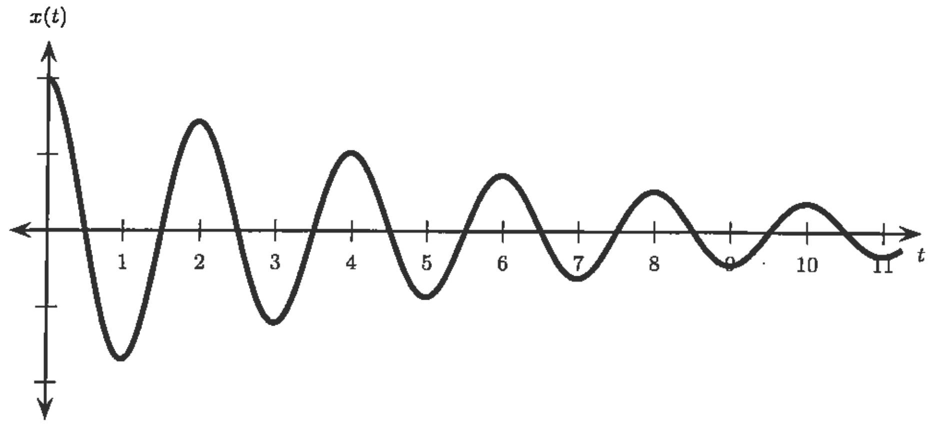
$$-\frac{\gamma T}{2} = \ln\left(\frac{C(t+T)}{C(t)}\right)$$

and this gives

$$\gamma = -\frac{2}{T}\ \ln\left(\frac{C(t+T)}{C(t)}\right) = \frac{2}{T}\ \ln\left(\frac{C(t)}{C(t+T)}\right)$$

3 Determining the damping constant

A graph of position vs. time for a lightly damped oscillator is shown. The time is measured in seconds.



Determine γ using the data from the graph.

Answer:

In general the amplitude satisfies

$$C(t) = C_0 e^{-\gamma t/2}$$

Then

$$\gamma = -\frac{2}{T} \ln\left(\frac{C(t+T)}{C(t)}\right)$$

Here, with $t = 0\, s$ *and* $T = 4\, s$, *inspection from the graph gives*

$$\frac{C(t+T)}{C(t)} = \frac{1}{2}.$$

Thus

$$\gamma = -\frac{2}{T} \ln\left(\frac{C(t+T)}{C(t)}\right) = -\frac{2}{4\, s} \ln\left(\frac{1}{2}\right) = 0.35\, s^{-1}.$$