Phys 230 2013

Lecture 26

Thurs: Physics seminar

Fri: Read

Mon: HW due

General oscillating systems:

The spring and mass system provides a typical example of an oscillating physical system. If we use x to represent the displacement away from equilibrium, then Newton's 2nd Law results in the equation of motion

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which is an example of the more general equation of motion for a simple harmonic oscillator

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

We note that ω is the angular frequency of oscillation, and the general solution to this equation is:

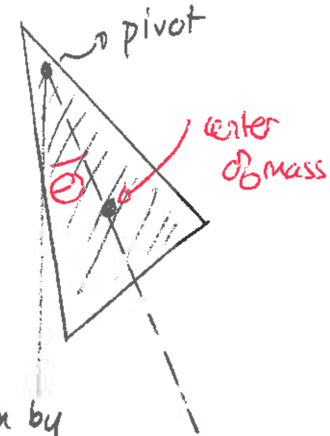
$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

or $x(t) = C\cos(\omega t + \phi)$

where A, B, C, ϕ are determined from initial conditions.

We shall see that this equation of motion arises in a wide range of situations. Generally what varies between these situations is the angular frequency and what stays the same is the form of the general solution.

The physical pendulum



A physical pendulum is a rigid object which can swing about a fixed pivot point. We can assess the dynamics of this via two ways:

i) rotational version of Newtonian dynamics

ii) energy

In both cases we describe the configuration of this system by

$$\boxed{\theta = \text{angular displacement of center of mass from vertical}}$$

Then the rotational dynamics approach starts with Newton's 2^{nd} Law

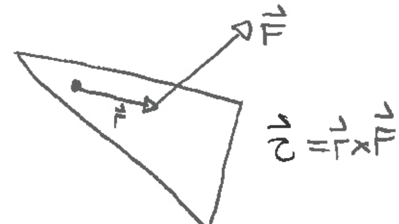
$$\vec{\tau}_{net} = I\vec{\alpha}$$

where $\vec{\alpha}$ = angular acceleration

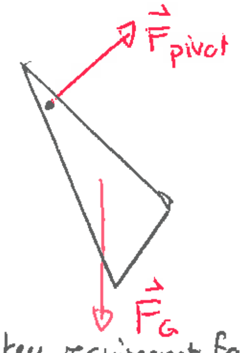
I = moment of inertia about pivot point

$\vec{\tau}_{net}$ = net torque about pivot point.

Recall these mean

angular acceleration	moment of inertia	torque
$\alpha = \dfrac{d^2\theta}{dt^2}$ direction given by r.h.r. here out of page.	 $I = \sum r^2 dm$ $I = \int r^2 dm$	 $\vec{\tau} = \vec{r} \times \vec{F}$ $\vec{r}$ = vector from pivot to application pt.

Quiz 1 || 1/7 →



Here there are two forces. The pivot force provides zero torque. The gravitational force produces a torque that tends to restore the system to equilibrium. This is a key requirement for oscillations

A detailed analysis results in the equation of motion:

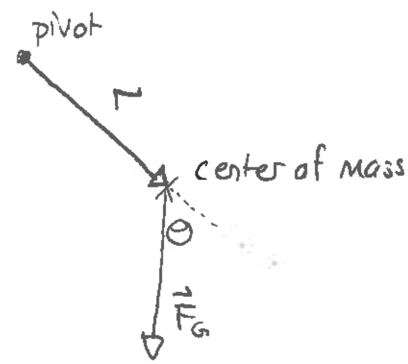
$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{Lmg}{I}\sin\theta} \qquad -(1)$$

where L = distance from pivot point to center of mass and m = total mass of pendulum.

Proof: Starting with $\vec{\tau}_{net} = I\vec{\alpha}$ we have

$$\vec{\tau}_{net} = \vec{\tau}_{grav} + \cancel{\vec{\tau}_{pivot}}$$

$$\Rightarrow \quad \vec{\tau}_{grav} = I\vec{\alpha}$$



Now $\vec{\tau}_{grav} = \vec{r} \times \vec{F}_G$ has direction into page and magnitude

$$rF_G\sin\theta = rmg\sin\theta$$
$$= Lmg\sin\theta$$

where L is the distance from the pivot to the c.o.m. The angular acceleration has magnitude $\frac{d^2\theta}{dt^2}$ and is out of the page. Take out as positive, $\vec{\tau}_{net} = I\vec{\alpha}$ gives:

$$I\frac{d^2\theta}{dt^2} = -Lmg\sin\theta$$

and that yields the above result ∎

Quiz 2: $\frac{0}{7} \rightarrow 7/7$

We see that (1) does not have the form $\frac{d^2\theta}{dt^2} = -\omega^2\theta$ and so it is not exactly the equation of motion for a simple harmonic oscillator. We attempt an approximation for $\sin\theta$ when θ is very small. To do so, the Taylor series for $\sin\theta$ is:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Thus if $\theta \ll 1$ then $\sin\theta \approx \theta$, so

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{Lmg}{I}\,\theta} \qquad (2)$$

This is exactly the equation of motion for a simple harmonic oscillator

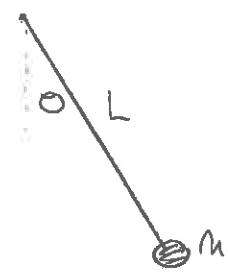
Quiz 3 $\frac{7}{7}$

We get

$$\boxed{\omega = \sqrt{\frac{Lmg}{I}}}$$

For a simple pendulum, a mass at the end of a string of length L, $I = mL^2$ and so

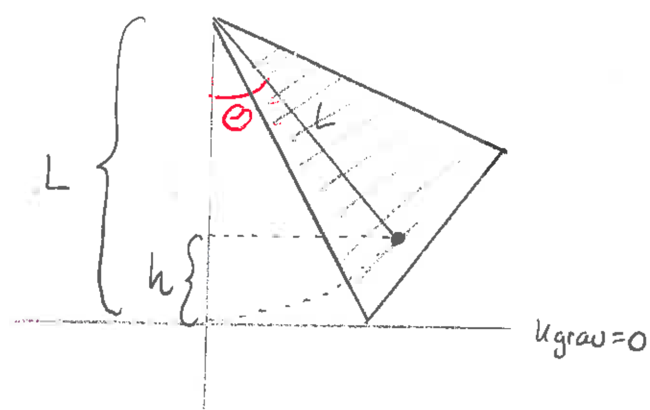
$$\boxed{\text{For a simple pendulum} \quad \omega = \sqrt{g/L}}$$



We can also analyze this from an energy perspective. For the physical pendulum

$$E = K_{rot} + U_{grav}$$

Set the $U_{grav} = 0$ level at the center of mass low point. In the indicated configuration,



$$K_{rot} = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2$$

$$U_{grav} = mgh = mg(L - L\cos\theta) = mgL(1-\cos\theta)$$

$$\Rightarrow \quad E = \frac{1}{2} I \left[\left(\frac{d\theta}{dt}\right)^2 + \frac{2mgL}{I}(1-\cos\theta)\right]$$

Again, this is not of the form for a harmonic oscillator, i.e.

$$E = \alpha \left[\left(\frac{d\theta}{dt}\right)^2 + \omega^2 \theta^2 \right]$$

but with $\cos\theta = 1 - \theta^2/2! + \theta^4/4! \ldots$ we get

$$\cos\theta \approx 1 - \theta^2/2!$$

and for $\theta \ll 1$

$$E = \frac{1}{2} I \left[\left(\frac{d\theta}{dt}\right)^2 + \frac{mgL}{I}\theta^2 \right]$$

giving $\omega = \sqrt{\frac{mgL}{I}}$

To summarize, for a physical pendulum and $\theta \ll 1$

$\frac{d^2\theta}{dt^2} = -\omega^2 \theta$ AND $E = \frac{1}{2} I \left[\left(\frac{d\theta}{dt}\right)^2 + \omega^2 \theta^2 \right]$

with $\omega = \sqrt{\frac{mgL}{I}}$ are both identical to the equation of motion and the energy equation for a simple harmonic oscillator:

$\frac{d^2x}{dt^2} = -\omega^2 x$ $E = \alpha \left[\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 \right]$

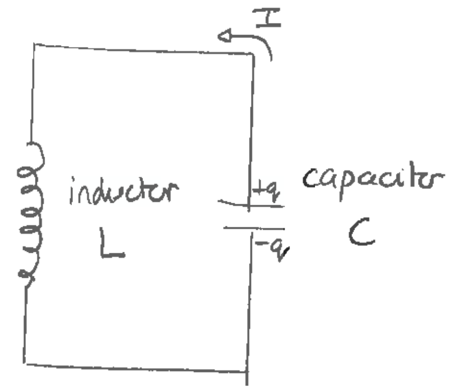
and the general solutions have identical form.

$$\theta = A\cos(\omega t) + B\sin(\omega t)$$

$$\theta = C\cos(\omega t + \phi)$$

Electrical Oscillators.

A simple circuit consisting of a capacitor and an inductor can produce oscillating charges, currents and voltages.



A capacitor can store charge $\pm q$ on its plates. When connected in a circuit the stored charge results in a discharging current

$$I = \frac{dq}{dt}$$

The inductor produces a "back" EMF or voltage. This resembles a form of inertia The back EMF is

$$V = -L\frac{dI}{dt}.$$

We can assess this from an energy perspective. The energy in the capacitor is

$$E_C = \frac{1}{2}\frac{1}{C}q^2$$

and the energy in the inductor is

$$E_L = \frac{1}{2}LI^2$$

So the total energy is:

$$E = E_L + E_I = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\frac{1}{C}q^2$$

Quiz 4

Here $E = \frac{1}{2}L\left[\left(\frac{dq}{dt}\right)^2 + \frac{1}{LC}q^2\right]$ so $\omega^2 = \frac{1}{LC}$. Thus

$$\omega = \frac{1}{\sqrt{LC}}.$$