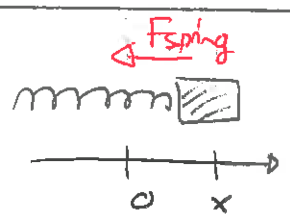
HW due today

Weds: Read 1.3, 1.4

Thurs: seminar

Simple Harmonic Motion

A standard example of simple harmonic motion is a block of mass m attached to a spring with spring constant k. The behavior of this system can be described using Newtonian physics and leads to the equation of motion for a simple harmonic oscillator:



Newton's 2nd Law:

$\vec{F}_{net} = m\vec{a}$

$\rightsquigarrow$

Equation of motion:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$\omega = \sqrt{k/m}$

$\rightsquigarrow$

Solution to equation of motion

$x(t) = A\cos(\omega t)$

This is the usual path of analysis for oscillatory systems. Note that the first step in the analysis of oscillatory systems often yields the same equation of motion:

$$\frac{d^2x}{dt^2} = -\alpha x$$

where α is a positive constant. This immediately yields the angular frequency via

$$\omega^2 = \alpha$$

So already, at the level of the equation of motion, we have some useful information. The second step, solving the equation of motion, is always the same for this type of differential equation. If we need specific information about the configuration variable (e.g. position) at later times we need this solution, but in many cases we do not actually need to solve the equation of motion.

Solving the equation of motion.

We have seen that one possible solution to the equation of motion is

$$x(t) = A\cos(\omega t) \qquad -(1)$$

where $\omega = \sqrt{k/m}$. We can use this to assess velocity, etc,.... For example

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t)$$

in this case. We now ask:

i) given this solution, how can we determine A

ii) is this the only type of solution?

Slide 1

We see that for the solution (1),

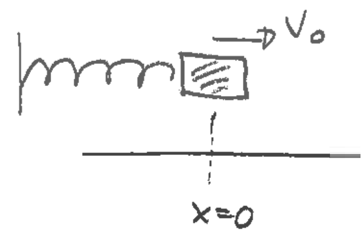
$$x(0) = A$$

$$v(0) = 0$$

This at least gives a meaning to the quantity A. However, this is only relevant if the initial velocity is zero, for example, if the block is released from rest. This will not necessarily be true

But suppose that at $t=0$ the block is at equilibrium and is given a sharp kick to the right Then $x(0)=0$ and $v(0) \doteq V_0 > 0$

The same equation of motion must apply but the solution of Eq (1) cannot apply to this situation. We need other solutions.



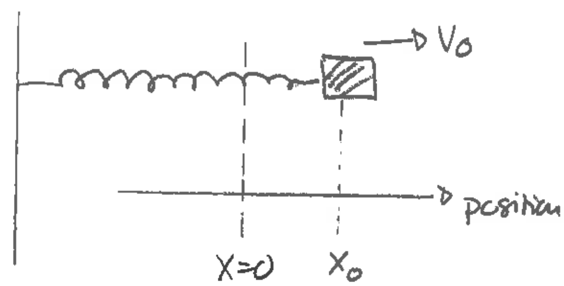
We can see by inspection that another possible solution is

$$x(t) = B \sin(\omega t) \qquad -(2)$$

Exercise: Check that (2) is a solution to the equation of motion for a simple harmonic oscillator.

For the solution of (2) we have $x(0)=0$ and $v(t) = \omega B \neq 0$. This will fit the above situation.

But now suppose that at $t=0$ the block is not at equilibrium and is given a sharp kick right. Then $x(0) \neq 0$ and $v(0) \neq 0$. So neither solution works, to describe this. We can try a combination:



$$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t).} \qquad -(3)$$

Exercise: Check that (3) is a solution to the equation of motion for a simple harmonic oscillator.

Note that the solution of equation (3) is a superposition or linear combination of the solutions of eqns (1) and (2). The ability to construct solutions by forming superpositions is a feature of differential equations of this type

<u>Exercise</u>: Show that if

$$x = x_1(t)$$
$$x = x_2(t)$$

each satisfy

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad -(A)$$

then

$$x = a_1 x_1(t) + a_2 x_2(t)$$

also satisfies eqn A provided that a_1, a_2 are constants.

We can now check whether (3) describes all of the previous cases.

Quiz 1 $4/6 \rightarrow 6/6$

Quiz 2 $3/6 \rightarrow 6/6$

We see that

If $x(0) = x_0$ and $v(0) = 0$ then the solution to the equation of motion is: $$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$$

Thus eqn (3) covers all possible solutions.

<u>General solution to the equation of motion.</u>

We have that the general solution to the equation of motion is

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

where A, B are constants

We can now ask, what is the amplitude of oscillation for this system? It cannot be A since if $B \neq 0$ the oscillator is moving at $t=0$ and could result in a position further from equilibrium. We need a more compact form for the solution. But we can show that an alternative form of solution is

$$\boxed{x(t) = C\cos(\omega t + \phi)} \qquad - \quad (4)$$

where $C =$ amplitude of oscillation (max displacement)

$\phi =$ phase of oscillation (sets time at which max displacement first occurs).

<u>Proof:</u> Suppose $x(t) = C\cos(\omega t + \phi)$

$$= C\cos(\omega t)\cos\phi - C\sin(\omega t)\sin\phi$$

So $\left.\begin{aligned} C\cos\phi &= A \\ -C\sin\phi &= B \end{aligned}\right\}$ for soln (3)

It follows that $\dfrac{\sin\phi}{\cos\phi} = -B/A \Rightarrow \phi = \tan^{-1}(-B/A)$

determines ϕ. Then $A^2 + B^2 = C^2\cos^2\phi + C^2\sin^2\phi = C^2$

$$\Rightarrow C = \sqrt{A^2+B^2}$$

determines C.

Energy in Oscillations

Consider the spring / block. The total mechanical energy is

$$E = K + U$$

and following the argument of the text:

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \left\{ \left(\frac{dx}{dt}\right)^2 + \frac{k}{m} x^2 \right\} = \frac{1}{2} m \left\{ \left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 \right\}$$

which gives the general form for the energy of a simple harmonic oscillator:

> For a simple harmonic oscillator, the energy is:
>
> $$E = \alpha \left\{ \left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 \right\}$$
>
> is conserved throughout the motion.

Proof: $$\frac{dE}{dt} = \alpha \left\{ \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 + \omega^2 \frac{d}{dt}(x^2) \right\}$$

$$= \alpha \left\{ 2 \frac{dx}{dt} \frac{d^2x}{dt^2} + 2\omega^2 x \frac{dx}{dt} \right\}$$

$$= 2\alpha \frac{dx}{dt} \left\{ \frac{d^2x}{dt^2} + \omega^2 x \right\}$$

But $\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \frac{dE}{dt} = 0$

Quiz 3 $\frac{2}{6}$

Quiz 4