Intermediate Dynamics: Group Exercises 8

Complex Numbers and Oscillations

1 Complex Functions and Simple Harmonic Motion

a) Let $z(t) = Ae^{i\omega t}$ where A is any real number. Show that the real part of this is a solution to the equation of motion for simple harmonic motion. Repeat this for the imaginary part.

b) Express $z(t) = Ae^{i\omega t}$ in terms of real and imaginary parts and use this to determine an expression for $\frac{dz}{dt}$. Use this and the rules of complex algebra to show that

$$\frac{dz}{dt} = i\omega A e^{i\omega t} = i\omega z(t).$$

Generalize this to the case where A is replaced by any complex number, D.

2 Damped Driven Oscillator

The equation of motion for a damped driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\,\cos{(\omega t)}.$$

a) Suggest possible complex functions $g(t)$ such that $\mathrm{Re}[g(t)] = \cos{(\omega t)}$. Find a simple function of this type in which trigonometric functions do not appear.

b) Consider

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m}\,e^{i\omega t}. \tag{1}$$

Assume that the solution to this has form

$$z(t) = De^{ut}$$

where D and u are complex constants. Substitute this into Eq. (1) and find algebraic expressions for u and D,

c) The complex number D can be represented in the form

$$D = Ae^{-i\delta}$$

where A and δ are real. Using this, determine an expression for

$$x(t) = \mathrm{Re}[z(t)]$$

and verify that A is the amplitude of oscillation.

d) Show that

$$Ae^{-i\delta} = \frac{F_0}{m} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}.$$

Using the fact that $A = |Ae^{-i\delta}|$ find an expression for A in terms of $F_0, m, \omega, \omega_0, \gamma$.

e) Determine an expression for the phase δ. *(Hint: Use the fact that* $1/z = z^*/|z|^2$*.)*