Phys 230
Fall 2013

Intermediate Dynamics: Homework 18

Due: 25 November 2013

1 King, *Vibrations and Waves*, 6.1, page 158.

2 King, *Vibrations and Waves*, 6.5, page 158.

3 King, *Vibrations and Waves*, 6.10, page 159.

4 Vibrating string with one end fixed and one free end.

Consider a string of length L which is stretched horizontally and whose left end is fixed to a support as illustrated in Fig. 1. The string's right end is connected to a support via a loop of string which can slide up and down the support without any friction.

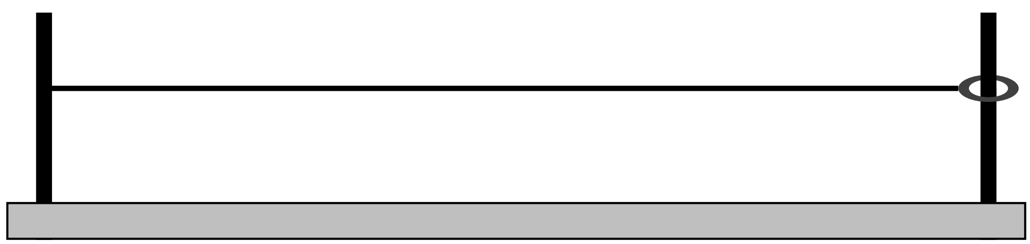

Figure 1: Question 4

The string obeys the same wave equation as that which has both ends fixed:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2 y}{\partial t^2}$$

where $y(x,t)$ gives the position of points along string above the horizontal equilibrium

The purpose of this exercise is to apply the boundary conditions to the problem and find the frequencies of the normal modes. It emerges that the appropriate boundary condition for the right hand end of the string is:

$$\left.\frac{\partial y}{\partial x}\right|_{x=L} = 0.$$

a) *(Challenging but not essential for progress with the rest of the problem.)* Justify the boundary condition at $x = L$ based on a consideration of the forces that the support exerts on the string.

b) The most general equation for a normal mode of the vibrating string initially at rest is:

$$y(x,t) = A\sin(kx+\alpha)\cos(\omega t)$$

where $k = \omega/v$ (note that the only difference compared to the equations in French is the appearance of the constant α). Apply the boundary conditions, determine α and find expressions for the frequencies of the normal and expressions for the normal modes solutions.

c) Plot the normal modes (i.e. set $t = 0$) for $n = 1, 2, 3$.

d) Compare the normal mode frequencies for this case to that where the string is fixed at both ends. Do this for $n = 1, 2, 3$.

5 Superpositions of traveling waves

Consider two traveling waves with possibly different amplitudes,

$$y_1(x,t) = A\cos(\omega t - kx)$$
$$y_2(x,t) = B\cos(\omega t + kx)$$

where $A \geqslant B > 0$.

a) The expressions for $y_1(x,t)$ and $y_2(x,t)$ differ from the standard expressions for traveling waves (this is to make the mathematics in the next sections simpler). Describe the directions of propagation of these waves.

In the following questions consider the superposition

$$y(x,t) = y_1(x,t) + y_2(x,t).$$

The complex exponential representations of these waves are:

$$z_1(x,t) = Ae^{i(\omega t - kx)}$$
$$z_2(x,t) = Be^{i(\omega t + kx)}.$$

b) Using the complex representations, show that the superposition has a complex representation of the form $z(x,t) = De^{i\omega t}$ where D is a complex number that depends on x but not t. Find an expression for D. What does this imply about the frequency of oscillation at each location?

c) Determine the amplitude of oscillation at each location. Find locations where the amplitude is a minimum and show that the minimum amplitudes are $A - B$.

d) *(Optional)* Show that $y(x,t) = C(x)\cos(\omega t + \phi(x))$ where $C(x)$ and $\phi(x)$ depend on position. Find an expression for $\phi(x)$ in terms of A, B and k.

e) Suppose that $A = B$. Show that $y(x,t)$ represents a standing wave. Is this a standing wave for all values of k?