# Intermediate Dynamics: Homework 17

Due: 20 November 2013

**1** King, *Vibrations and Waves*, 5.10, page 134.

## 2 Complex exponential representations of waves and energy

Traveling sinusoidal waves on a string can be represented by:

$$y(x,t) = \begin{cases} A\cos(kx - \omega t + \phi) & \text{right moving wave} \\ A\cos(kx + \omega t + \phi) & \text{left moving wave.} \end{cases}$$

In many situations it is convenient to represent the wave by a complex function $z(x,t)$ such that $y(x,t) = \text{Re}[z(t)]$.

a) Show that the right and left moving solutions can be represented by $z(x,t) = De^{i(kx-\omega t)}$ and $z(x,t) = De^{i(kx+\omega t)}$ where $D = Ae^{i\phi}$ (i.e. show that $\text{Re}[z(t)]$ gives the correct real solution in each case).

b) Starting from the fact that the energy density is $\frac{1}{2}\,\mu\omega^2 A^2$, show that the average energy density per cycle of these waves is given by

$$\frac{1}{2}\mu\omega^2 |z(x,t)|^2$$

c) Show that the average power transmitted by these waves is given by

$$\overline{P} = \frac{1}{2}\mu\omega^2 v |z(x,t)|^2$$

where $v$ is the *wave velocity.* Describe whether this is positive or negative for left or right moving waves.

## 3 Energy of a wave pulse

The general solution to the wave equation can be constructed as a combination of left and right moving disturbances

$$y(x,t) = f(x - vt) \qquad \text{right moving} \tag{1}$$

$$y(x,t) = g(x + vt) \qquad \text{left moving} \tag{2}$$

where $f(u)$ and $g(u)$ are any sufficiently well behaved functions of $u$.

Consider waves on a string of infinite extent and with mass per unit length $\mu$.

a) Show that the energy density for *any left moving disturbance* is

$$\mu v^2 \left(\frac{dg}{du}\right)^2 \bigg|_{u=x+vt}.$$

*Note: The vertical bar means that you should first differentiate $g(u)$ with respect to $u$ and then substitute $u = x + vt$ into the result.*

In the following parts, suppose that a left moving disturbance is described by

$$g(u) = Ae^{-u^2/2}.$$

b) Determine an expression for the energy density as a function of $x$ and $t$.

c) Determine the total energy in the string at any time. *Hint: this will require an integral of a Gaussian-type function over the entire real line. This can be done by looking up the integral or using a calculator/MAPLE/Wolfram to evaluated it; you may want to perform a u substitution before evaluating.*

d) Plot the energy density as a function of $x$ at $t = 0$.

e) Plot the energy density as a function of $x$ at $t = 3/v$.