Intermediate Dynamics: Homework 16

Due: 18 November 2013

1 King, *Vibrations and Waves*, 5.2, page 133.

2 King, *Vibrations and Waves*, 5.8a), page 134. Only do part a).

3 Traveling wave pulses on a string.

Suppose that a pulse on a string is represented by

$$y(x,t) = Ae^{-(x-vt)^2/a^2}$$

where A, v and a are all positive constants.

a) Show, by explicit substitution, that $y(x,t)$ is a solution to the wave equation.

b) Determine an expression for the transverse velocity of any point along the string at any time t. Use this to determine an expression for the the transverse velocity of any point along the string at any time $t = 0$. Describe the ranges for which this is positive, negative and zero. Does your description agree with what you would expect from physical considerations?

Now suppose that the pulse on the string has the form

$$y(x,t) = \begin{cases} 2 - 2(x-vt)^2 \quad \text{if} \quad vt - 1 \leqslant x \leqslant vt + 1 \quad \text{and} \\ 0 \quad \text{otherwise,} \end{cases}$$

where $v = 3$.

c) Plot a graph of the string profile at $t = 0$.

d) Plot a graph of the string profile at $t = 1$.

4 Complex solutions to the wave equation

The wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2 y}{\partial t^2} \tag{1}$$

describes the displacement of the wave medium (e.g string), $y(x,t)$, which is necessarily a real quantity. Typical sinusoidal wave solutions to this are

$$\begin{aligned} y(x,t) &= C\cos(kx - \omega t \pm \phi) \quad \text{or} \\ y(x,t) &= C\cos(kx + \omega t \pm \phi) \end{aligned} \tag{2}$$

where the amplitude $C > 0$ and phase ϕ are real constants. However, it can be useful to consider an equivalent equation for a complex valued function $z(x,t)$, i.e.

$$\frac{\partial^2 z}{\partial t^2} = v^2 \frac{\partial^2 z}{\partial x^2}. \tag{3}$$

a) Show explicitly by direct substituting into Eq. (3) that each of the complex exponential expressions

$$\begin{aligned} z(x,t) &= De^{i(kx+\omega t)} \\ z(x,t) &= De^{i(kx-\omega t)} \\ z(x,t) &= De^{-i(kx+\omega t)} \\ z(x,t) &= De^{-i(kx-\omega t)} \end{aligned}$$

where D is any complex constant, are solutions to the complex version of the wave equation provided that ω and k satisfy an appropriate condition. Find the condition.

b) Suppose that in the above expressions $D = Ce^{i\phi}$. Then show that each of the four possibilities produces one of the real solutions of Eq. (2) via $y(x,t) = \text{Re}[z(x,t)]$.

c) Which of the complex exponential expressions correspond to left moving and which to right moving solutions.

d) In a particular situation, the complex exponential description for a wave on a string is $z(x,t) = (12 + i5)\ e^{i(\pi x+2\pi t)}$. Determine the real solution $y(x,t)$ and its amplitude, wavelength, frequency and wave speed.

e) In another situation, the complex exponential description for a wave on a string is $z(x,t) = (3-i4)\ e^{i(2\pi x+10\pi t)}$. Determine the maximum transverse velocity of any segment of the string. Does this depend on the location of the segment? Is the maximum transverse velocity equal to the wavespeed?

5 King, *Vibrations and Waves*, 5.15, page 135. One crucial assumption is that a is much larger than $y_r - y_{r-1}$ and $y_r - y_{r+1}$.