Intermediate Dynamics: Homework 12

Due: 4 November 2013

1 RLC circuits

An RLC circuit is constructed using an inductor for which $L = 8.2 \times 10^{-3}\,\mathrm{H}$ and $C = 100 \times 10^{-6}\,\mathrm{F}$ (these are the values on the PASCO RLC circuit board).

a) Determine the range of values of R for which the circuit will produce lightly damped oscillations. Determine the range of values of R for which the circuit will produce heavy damping (overdamped motion).

b) Determine ω_0.

c) For $R = 10\,\Omega$ determine ω and Q.

d) For $R = 1\,\Omega$ determine ω and Q.

Now consider an RLC circuit for which $L = 100\,\mathrm{mH}$ and $C = 2 \times 10^{-6}\,\mathrm{F}$ and $R = 3\,\Omega$

e) Determine ω_0 and determine whether the circuit is heavily damped or displays damped oscillations.

f) Suppose that the initial energy in the circuit is $1.25 \times 10^{-3}\,\mathrm{J}$. Determine the time taken for the energy in the circuit to be reduced to 1/2 of its initial value. Repeat for 1/4 of its initial value.

2 King, *Vibrations and Waves*, 3.1, page 74.

3 Driven damped oscillator: amplitude and force

The amplitude of a damped driven oscillator is given by

$$A(\omega) = \frac{F_0}{m} \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}.$$

In many cases it is convenient to rephrase the amplitude and various other physical quantities in terms of a dimensionless quantity that represents the frequency of oscillation. Let $u = \omega/\omega_0$.

a) Show that

$$A = \frac{F_0}{m\omega_0^2} \frac{1}{u} \frac{1}{\sqrt{(1/u - u)^2 + 1/Q^2}}.$$

b) Using this determine the value of ω at which the amplitude is maximum and determine the maximum amplitude.

c) Determine the ratio of amplitudes $A_{\max}/A(0)$ where $A_{\max}$ is the maximum amplitude. What information does the quality factor provide about this amplification?

d) Is the maximum amplitude equal to the amplitude when $\omega = \omega_0$ in general? What about when $Q \gg 1$?

e) Show that the amplitude of the net force acting on the oscillator is

$$F_0 \frac{u}{\sqrt{(1/u - u)^2 + 1/Q^2}}.$$

f) Show that the amplitude of the net force is largest when $\omega = \omega_0/\sqrt{1 - 1/2Q^2}$ and show that the maximum amplitude is $F_0 Q/\sqrt{1 - 1/4Q^2}$.

g) Describe whether this maximum force is attained for all values of the damping constant and natural frequencies.

4 Transient and steady state solutions

The equation of motion for a damped driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \tag{1}$$

The transient solution x_t satisfies

$$\frac{d^2x_t}{dt^2} + \gamma\frac{dx_t}{dt} + \omega_0^2 x_t = 0$$

and the steady state solution satisfies

$$\frac{d^2x_s}{dt^2} + \gamma\frac{dx_s}{dt} + \omega_0^2 x_s = \frac{F_0}{m}\cos\omega t$$

For lightly damped oscillations the transient solution

a) Show that $x(t) = x_t(t) + x_s(t)$ also satisfies Eq. (1).

The solutions to these are

$$\begin{aligned} x_s &= A(\omega)\cos(\omega t - \delta) \\ x_t &= Ce^{-\gamma t/2}\ \cos(\omega_0 t + \phi) \end{aligned}$$

where $A(\omega)$ and δ are as given in class.

b) Briefly describe how you might obtain values for the constants C and ϕ.