Intermediate Dynamics: Class Exam II

22 October 2012

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Instructions

- There are 5 questions on 5 pages.
- Show your reasoning and calculations and always justify your answers.

Physical constants and useful formulae

$$U_{\text{spring}} = \frac{1}{2}\,k(\Delta s)^2$$
$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A \sin B$$

Question 1

The motion of an oscillator is represented by the complex function

$$z(t) = (3+3i)\;e^{i4\pi t}$$

a) Determine an expression for the position of the oscillator $x(t)$.

$x(t) = \mathrm{Re}[z(t)]$

$z(t) = (3+3i)[\cos 4\pi t + i \sin 4\pi t]$

$= [3\cos 4\pi t - 3\sin 4\pi t] + i[3\cos 4\pi t + 3\sin 4\pi t]$

$x(t) = 3\cos 4\pi t - 3\sin 4\pi t$

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b) Determine the velocity of the oscillator at $t = 0$.

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$v(t) = \frac{dx}{dt} = -12\pi \sin 4\pi t - 12\pi \cos 4\pi t$

at $t=0$ $v = -12\pi$

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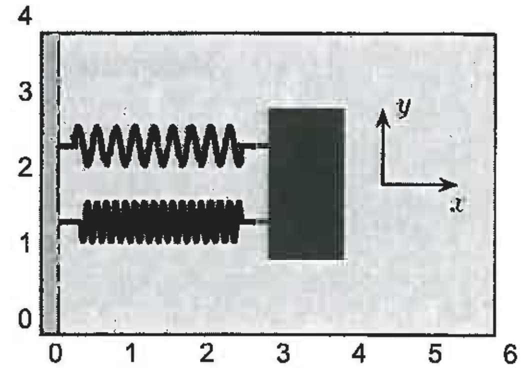
Question 2

A block of mass m is connected to two springs, each of which has the same unstretched length. The block can slide left and right along a horizontal frictionless table. The block is displaced along the illustrated x direction and when it does this the two springs are stretched by the same length. The upper spring has spring constant k_1 and the lower spring k_2. Let x denote the displacement of the block from equilibrium.



a) Let x denote the displacement of the block from equilibrium. Determine an expression for the energy of the system in terms of x and $\frac{dx}{dt}$.

$$E = K + U_{spring\,1} + U_{spring\,2}$$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2$$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{k_1+k_2}{2}x^2$$

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b) Determine an expression for the angular frequency and the period of oscillation of the block.

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Need $E = \alpha\left\{\left(\frac{dx}{dt}\right)^2 + \underbrace{\omega_0^2}_{\to \text{angular frequency}}x^2\right\}$

Here

$$E = \frac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2 + \frac{k_1+k_2}{m}x^2\right\}$$

$$\Rightarrow \omega_0^2 = \frac{k_1+k_2}{m} \Rightarrow \boxed{\omega_0 = \sqrt{\frac{k_1+k_2}{m}}}$$

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$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k_1+k_2}}$$

Question 3

The charge in an RLC series circuit satisfies

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}\,q = 0$$

where L is the inductance, R the resistance and C the capacitance.

a) Determine expressions for the undamped (natural) angular frequency, ω_0, the damping constant γ and the quality factor, Q, in terms of R, L and C.

$$\frac{d^2q}{dt^2} + \underbrace{\left(\frac{R}{L}\right)}_{\gamma}\frac{dq}{dt} + \underbrace{\left(\frac{1}{LC}\right)}_{\omega_0^2} q^2 = 0$$

$$\Rightarrow \gamma = \frac{R}{L} \qquad Q = \frac{\omega_0}{\gamma} = \frac{L}{R}\frac{1}{\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

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b) Suppose that $L = 3.0 \times 10^{-4}\,$H, $C = 3.0 \times 10^{-8}\,$C and $R = 2.0\,\Omega$. If the total energy in the system is initially E_0, find the amount of time required for the total energy to reach $E_0/4$.

$$E = E_0 e^{-\gamma t}$$

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$$\frac{\cancel{E_0}}{4} = \cancel{E_0} e^{-\gamma t} \Rightarrow \frac{1}{4} = e^{-\gamma t} \Rightarrow 4 = e^{\gamma t}$$

$$\Rightarrow \ln 4 = \gamma t \Rightarrow t = \frac{1}{\gamma}\ln 4$$

$$= \frac{L}{R}\ln 4$$

$$= \frac{3.0\times10^{-4}H}{2.0\Omega}\ln 4 = 2.1\times10^{-4}s$$

Question 3 continued ...

c) The same system is driven by a time varying potential and the charge satisfies

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \cos(\omega t)$$

where ω is the driving frequency and $V_0 = 5.0\,\text{V}$. For what value of ω would the power absorbed would be a maximum? Determine range of angular frequencies over which the power would be at least half of this maximum value.

Max power when $\omega = \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{3.0\times10^{-4}\times 3.0\times10^{-8}}}$

$= \frac{1}{3.0}\times10^{6} = 3.3\times10^{5}\,\text{Hz}$

Range $=$ FWHM $= \gamma = \frac{R}{L} = \frac{2.0\Omega}{3.0\times10^{-4}} = 1.5\times10^{3}\,\text{Hz}$ 5

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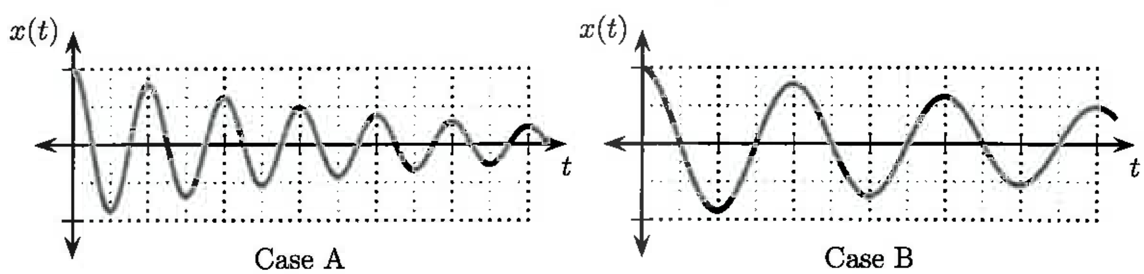
Question 4

Graphs of positions vs. time for two lightly damped oscillators are as illustrated.



a) Which of the following (choose one) is true regarding the damping constants of the oscillators?

i) $\gamma_A = \gamma_B$
ii) $\gamma_A > \gamma_B$
iii) $\gamma_A < \gamma_B$

$C(t) = C_0 e^{-\gamma t/2}$

Look for t when $C = \frac{C_0}{2} \Rightarrow \frac{1}{2} = e^{-\gamma t/2}$

larger t for B $\Rightarrow$ smaller γ for B.

b) Which of the following (choose one) is true regarding the quality factors of the oscillators?

i) $Q_A = Q_B$
ii) $Q_A > Q_B$
iii) $Q_A < Q_B$

For B $\gamma_B = \frac{1}{2}\gamma_A$

Also $\omega_{0B} = \frac{1}{2}\omega_{0A}$

$\Rightarrow Q_B = Q_A$

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Question 5

The complex equation of motion for a undamped driven oscillator is

$$\frac{d^2z}{dt^2} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

where ω_0 is the natural frequency and ω is the driving frequency. Assume that the solution has the form $z = De^{ut}$ where D and u are complex constants. Find expressions for D, u and the amplitude of oscillation in terms of ω, ω_0, F_0 and m.

$$u^2 De^{ut} + \omega_0^2 De^{ut} = \frac{F_0}{m} e^{i\omega t}$$

$$\Rightarrow \boxed{u = i\omega}$$

Then $$u^2 D + \omega_0^2 D = \frac{F_0}{m}$$

$$\Rightarrow D(\omega_0^2 - \omega^2) = \frac{F_0}{m} \quad \Rightarrow D = \frac{F_0}{m} \frac{1}{\omega_0^2 - \omega^2}$$

Finally amplitude $= |D| = \frac{F_0}{m} \frac{1}{|\omega_0^2 - \omega^2|}$

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