Intermediate Dynamics: Final Exam

10 December 2012

Name: Solution **Total:** /70

Instructions

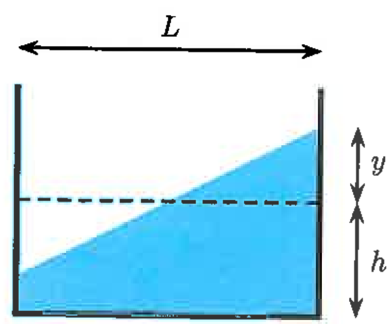
- There are 8 questions on 9 pages.
- Show your reasoning and calculations and always justify your answers.

Physical constants and useful formulae

Speed of light $c = 3.0 \times 10^8$ m/s

Question 1

A container holds a fluid of density ρ. The dashed line indicates the level of the fluid when it is at equilibrium and the width of the container is b. The other parameters are illustrated in the diagram. The fluid can be made to slosh back and forth so that, at any instant, its cross sectional profile is as illustrated. It can be shown that the kinetic energy is $K = \frac{b\rho L^3}{60h}\left(\frac{\partial y}{\partial t}\right)^2$ and the potential energy is $U = \frac{1}{6}\, b\rho g L y^2$. Use these to determine an expression for the frequency of oscillation of the fluid.



$$E = K + U = \frac{b\rho L^3}{60h}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{6} b\rho g L y^2$$

$$E = \frac{b\rho L^3}{60h}\left\{\left(\frac{\partial y}{\partial t}\right)^2 + \underbrace{\frac{10gh}{L^2}}_{\omega^2} y^2\right\}$$

/5

$$\Rightarrow \quad \omega = \sqrt{\frac{10gh}{L^2}}$$

Question 2

The complex version of the equation of motion for a damped driven harmonic oscillator is

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

where γ is the damping constant and ω_0 is the natural frequency of oscillation.

a) Assume that the solution to the equation of motion is $z(t) = De^{ut}$ where u and D are complex constants. Determine an expression for u in terms of γ and ω_0.

$$\frac{dz}{dt} = uz \Rightarrow \frac{d^2z}{dt^2} = u^2 z$$

$$\Rightarrow \left[u^2 + \gamma u + \omega_0^2\right] z = 0 \Rightarrow u^2 + \gamma u + \omega_0^2 = 0$$

$$u = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

(3)

b) Show that if $\gamma < 2\omega_0$ this describes solutions which oscillate. Show that, if $D = Ce^{i\delta}$, then the real solution can be expressed in the form $x(t) = Ce^{-t/\tau}\cos(\omega t + \delta)$. Determine expressions for ω and τ in terms of γ and ω_0. (5)

Here $u = \frac{-\gamma \pm i\sqrt{4\omega_0^2 - \gamma^2}}{2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \gamma^2/4}$

take positive root an set $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$ ← 1

$$\Rightarrow z = Ce^{i\delta} e^{-\gamma t/2} e^{i\omega t}$$

$$= Ce^{-\gamma t/2} e^{i(\omega t + \delta)}$$

3 → $x = \text{Re } z = Ce^{-\gamma t/2}\cos(\omega t + \delta)$

1 → $\tau = 2/\gamma$

/8

Question 3

A string is stretched between two supports at $x = 0$ and $x = L$.. At each support the string can slide freely. Standing waves on the string satisfy the boundary conditions that

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial y}{\partial x}\right|_{x=L} = 0.$$

Which of the following (choose one) represents a standing wave solution that is consistent with these boundary conditions?

if sin(kx)

$\frac{\partial y}{\partial x} = -k\cos(kx) \neq 0$ at $x=0$

a) ~~$y(x,t) = A\cos(kx - \omega t)$.~~ traveling

b) $y(x,t) = A\cos(kx)\cos(\omega t)$ for certain special k.

c) $y(x,t) = A\cos(kx)\cos(\omega t)$ for any k. → satisfies $\left.\frac{\partial y}{\partial x}\right|_0 = 0$

d) ~~$y(x,t) = A\sin(kx)\cos(\omega t)$ for certain special k.~~

e) ~~$y(x,t) = A\sin(kx)\cos(\omega t)$ for any k.~~

satisfies $\left.\frac{\partial y}{\partial x}\right|_L = 0$ for certain k

/5

Question 4

The distance from the Earth to Andromeda galaxy, according to observers at rest with respect to Earth, is 2×10^6 lt-yr. According to these observers, a rocket ship carries an astronaut from the Earth to Andromeda at a speed of $0.8c$. Determine the time that the trip from Earth to Andromeda takes according to an observer at rest with respect to the Earth. Determine the time taken according to the astronaut.

Earth $v = \frac{\Delta x}{\Delta t} \Rightarrow \Delta t = \frac{2\times10^6\, c\cdot yr}{0.8c} = 2.5\times10^6 yr$

Time dilation

$$\underset{\substack{\text{both}\\\text{at same}\\\text{location}}}{\Delta t} = \underset{\substack{\text{diff}\\\text{locations}}}{\Delta t}\sqrt{1 - v^2/c^2} = \Delta t\sqrt{1-(0.8)^2} = 0.6\,\Delta t_{\substack{\text{diff}\\\text{locations}}}$$

Earth measures

$\Rightarrow \Delta t = 0.6 \times 2.5\times10^6 yr = 1.5\times10^6 yr.$

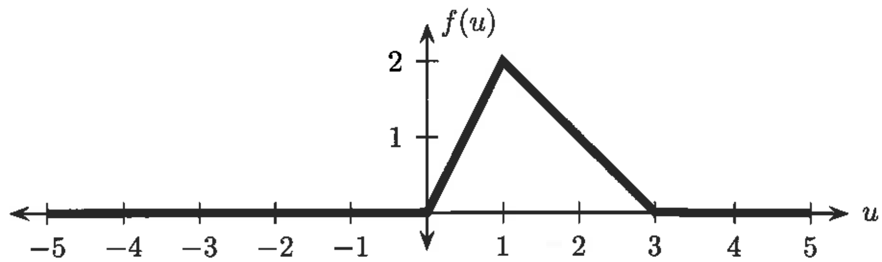
/6

Question 5

A tightly stretched string obeys the classical wave equation where $v = 2\,\mathrm{m/s}$. Consider the solution

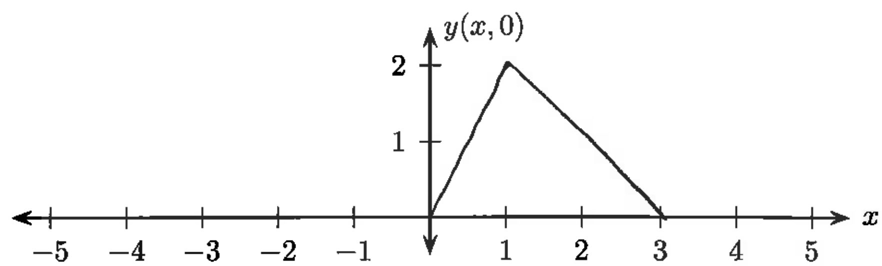
$$y(x,t) = f(x - vt)$$

where $f(u)$ is as plotted below (the horizontal units are meters).

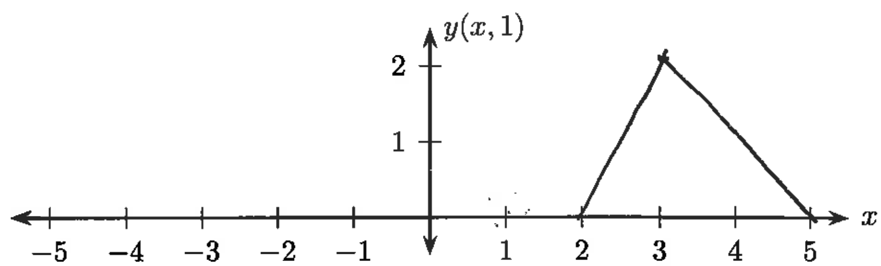


a) Using the axes below, plot snapshots of the string at $t = 0\,\mathrm{s}$ and $t = 1\,\mathrm{s}$

$y(x,0) = f(x)$



$y(x,1) = f(x-2)$



4

Question 5 continued . . .

b) Show that, for such a disturbance, the energy density at time t is given by

$$\mu v^2 \left(\frac{df}{du}\right)^2 \bigg|_{u=x-vt}.$$

$V = \sqrt{\frac{T}{\mu}}$

and use this to determine an expression for the total energy of the string at $t = 0\,\text{s}$. $\Rightarrow T = V^2\mu$

7

$$\text{Energy density} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = -v\frac{df}{du}$$

$$\frac{\partial y}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du}$$

$$\Rightarrow \text{energy density} = \frac{1}{2}\mu v^2\left(\frac{df}{du}\right)^2 + \frac{1}{2}\mu v^2\left(\frac{df}{du}\right)^2$$

$$= \mu v^2\left(\frac{df}{du}\right)^2 \longrightarrow = \begin{cases} 4\mu v^2 & 0<u<1 \\ \mu v^2 & 2<u<3 \end{cases}$$

The energy is $\displaystyle\int_{-\infty}^{\infty} \text{energy density}\, dx$

43

At $t=0$

$$\int_0^1 4\mu v^2\,dx + \int_1^3 \mu v^2\,dx = 6\mu v^2$$

/14

Question 6

An observer in a space station observes a pair of rotating binary stars. At one instant the stars are equally distant from the observer and star α is moving away with speed u while star β is moving toward the observer with speed u. At this instant each emits a bright pulse of light. Which of the following (choose one) is true?

c same in all frames

a) The light pulses will arrive at the same time at the observer.

b) The pulse from α arrives first since the light from it travels faster than that from β.

c) The pulse from α arrives first since the light from it travels slower than that from β.

d) Whether the pulse from α arrives first or second depends on whether the observer is moving toward or away from the pair of stars.

/4

Question 7

An observer is midway between the ends of a rocket ship, which has length 5 lt-sec = 5 c·sec. The left end of the rocket is equipped with a red light source and the right end with a blue light source. The rocket travels with speed $4c/5$ relative to an observer on a space station. The space station observer uses unprimed coordinates and the rocket observer uses primed coordinates. The two observers coincide in space when their clocks read $t = 0$ and $t' = 0$.

in rest frame

Each light produces a brief flash. Each of these flashes eventually arrive at the two observers. The space station observer records various data about the flashes and he concludes that *according to him the flashes were produced simultaneously at* $t = 0$ *s.*

2 a) Determine the length of the rocket according to the space station observer. Specify the location of each light source according to the space station observer at $t = 0$ s.

length contraction $d = D\sqrt{1 - v^2/c^2}$

$= 5c\cdot sec \sqrt{1-(4/5)^2} = 3c\cdot sec$

midpoint at 0 c·sec

red source −1.5c·sec

blue source 1.5c·sec

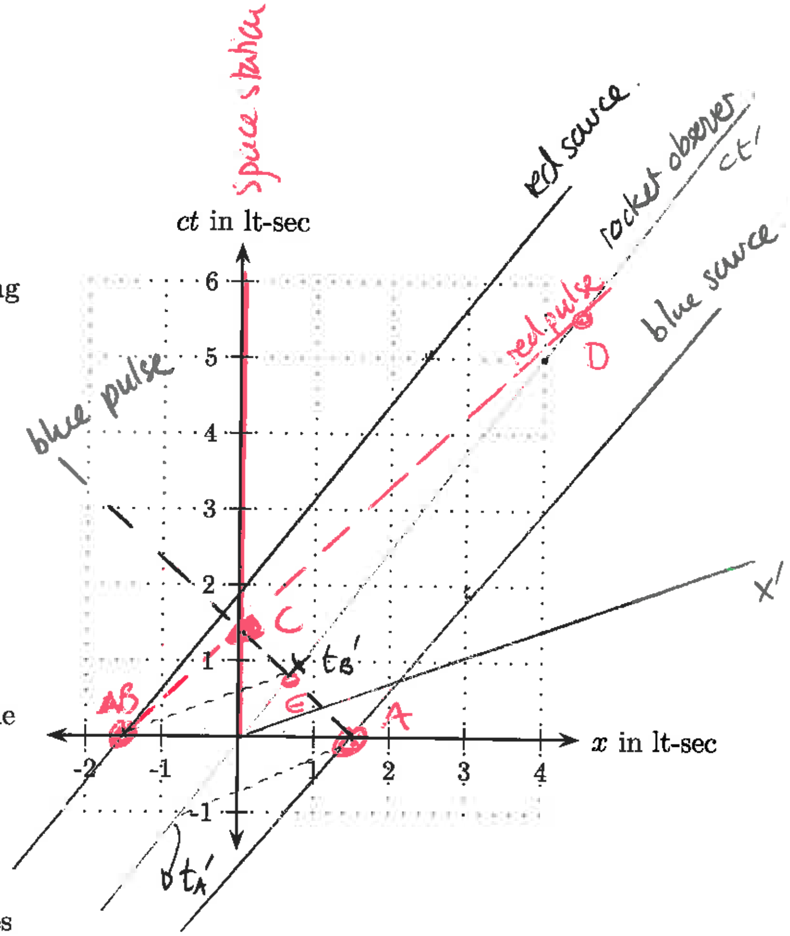
Question 7 continued ...

9 b) Sketch the *worldlines* of the following on the spacetime diagram:

- The space station observer.
- The rocket observer.
- The red light source.
- The blue light source.
- The red light pulse.
- The blue light pulse.

Sketch the following *events* on the spacetime diagram:

- A • Blue light pulse is produced.
- B • Red light pulse is produced.
- C • Red and blue light pulses arrives at the space station observer.



2 c) Using *the spacetime diagram* describe the order in which the light pulses are produced according to the rocket observer.

We can see t_A' and t_B' on the diagram.

Clearly t_A' is earlier → rocket observer states

blue produced first

red later

Question 7 continued . . .

3 d) Using equations, determine the times at which each light pulse is produced according to the rocket observers.

	Space station		Rocket	
	t	x	t'	x'
red	0 s	$-1.5\,c\cdot s$	2 s	
blue	0 s	$1.5\,c\cdot s$	-2 s	

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(4/5)^2}} = 5/3$$

$$= \frac{5}{3}\left(t - \frac{4}{5c}x\right)$$

For red $\quad t' = \frac{\cancel{5}}{3}\left(-\frac{4(-1.5\cancel{c}\cdot s)}{\cancel{5}\cancel{c}}\right) = 2\,s$

Blue

$t' = \frac{5}{3}\left(-\frac{4}{5c}\right)(1.5c\cdot s) = -2s$

3 e) The two observers record data about the arrival of the red and blue pulses *at the space station observer.* Which of the following (choose one) is true?

Lorentz transf

$t' = \gamma\left(t - \frac{v}{c^2}x\right)$

same both $\quad$ same both

same

i) Pulses arrive simultaneously according to both observers.

ii) Pulses arrive simultaneously according to space station observer. Red is first according to rocket observer.

iii) Pulses arrive simultaneously according to space station observer. Blue is first according to rocket observer.

iv) Red arrives first according to both observers.

v) Blue arrives first according to both observers.

3 f) The two observers record data about the arrival of the red and blue pulses *at the rocket observer.* Which of the following (choose one) is true?

red arrives D

blue " E

Spacetime diagram shows

$t_D > t_E$

$t_D' > t_E'$

i) Pulses arrive simultaneously according to both observers.

ii) Pulses arrive simultaneously according to rocket observer. Red is first according to space station observer.

iii) Pulses arrive simultaneously according to rocket observer. Blue is first according to space station observer.

iv) Red arrives first according to both observers.

v) Blue arrives first according to both observers.

/22

Question 8

A nucleus with rest mass 1875 MeV/c^2 is initially at rest. This is struck by a photon with momentum 3000 MeV/c. The photon is absorbed, leaving the nucleus in an excited state. Conservation rules give that the energy of the nucleus after the collision is 4875 MeV and the mass of the nucleus after collision is 3842 MeV/c^2. Determine an expression for the speed of the nucleus after collision.

momentum conservation

$\Rightarrow p_{\text{nucleus after}} = p_{\text{photon before}}$

$\Rightarrow p_{\text{nucleus after}} = 3000\,\text{MeV}/c$

$$p = \frac{mu}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \quad p^2\left(1-u^2/c^2\right) = m^2u^2$$

$$\Rightarrow \quad p^2 = \left[m^2 + p^2/c^2\right]u^2$$

$$\Rightarrow \quad u = \frac{p}{\sqrt{m^2+p^2/c^2}} = \frac{3000\,\text{MeV}/c}{(3842^2+3000^2)^{1/2}\,\text{MeV}/c^2}$$

$$= 0.62c$$

/6