# Qubit Channel Parameter Estimation with Suboptimal Resources

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## Abstract

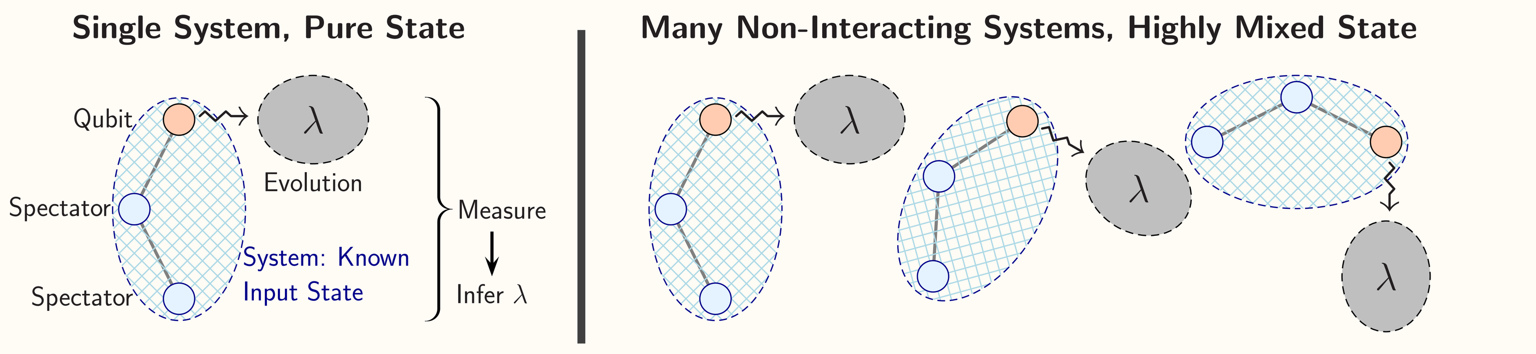
We consider physical protocols for estimating the parameter in a single-parameter dependent qubit channel whose type is known. The quantum Fisher information (QFI) can quantify the accuracy of an estimation protocol and, using this, optimal protocols for qubit channels are well known. These require pure input states and sometimes use entangled noiseless spectator qubits to enhance estimation accuracy.

We consider qubit channel parameter estimation in situations where optimal resources are not available. Specifically, we assume that every qubit involved is initially in a mixed state quantified by purity $0 \leqslant r \leqslant 1$. We assume that the channel is used once and ask whether involving additional spectator qubits with the same purity and subject to independent noisy evolution process can enhance the QFI.

We compare two protocols: one using a single qubit and a single channel invocation and the other using $n$ qubits whose input state is prepared from single qubit initial states using a particular correlating unitary. We compare these for the cases where the channel is invoked on one out of $n$ qubits and whenever the purity satisfies $r \ll 1/\sqrt{n}$. We show that for unital channels the correlated state protocol gives a roughly $n$-fold enhancement in estimation accuracy that is reduced by the noisy evolution of the spectator qubits. We describe how to assess when the additional noise eliminates any advantages of the correlating protocol.

## Physical Parameter Estimation

A qubit undergoes an evolution of known type dependent on a parameter, $\lambda$. The task is to **estimate the parameter with optimal accuracy.** Any estimation procedure requires a choice of system input state prior to the evolution, a post-evolution measurement and an estimator function that extracts a parameter estimate from measurement outcomes. Possible scenarios are illustrated.




In general quantum estimation has focused on questions such as whether spectator qubits, entanglement or other correlation can enhance estimation accuracy, quantified via the Cramér-Rao bound [1]:

$$\text{m.s.e}\,(\lambda_{\text{est}}) = \left\langle \left(\lambda_{\text{est}} - \lambda\right)^2 \right\rangle \geqslant \frac{1}{F(\lambda)} \geqslant \frac{1}{H(\lambda)}.$$

MSE depends on estimator choice.

Classical Fisher information depends on measurement choice, not estimator.

QFI depends on $\hat{\rho}_{\text{f}}$ but not measurement choice nor estimator.

The quantum Fisher information (QFI) is given by

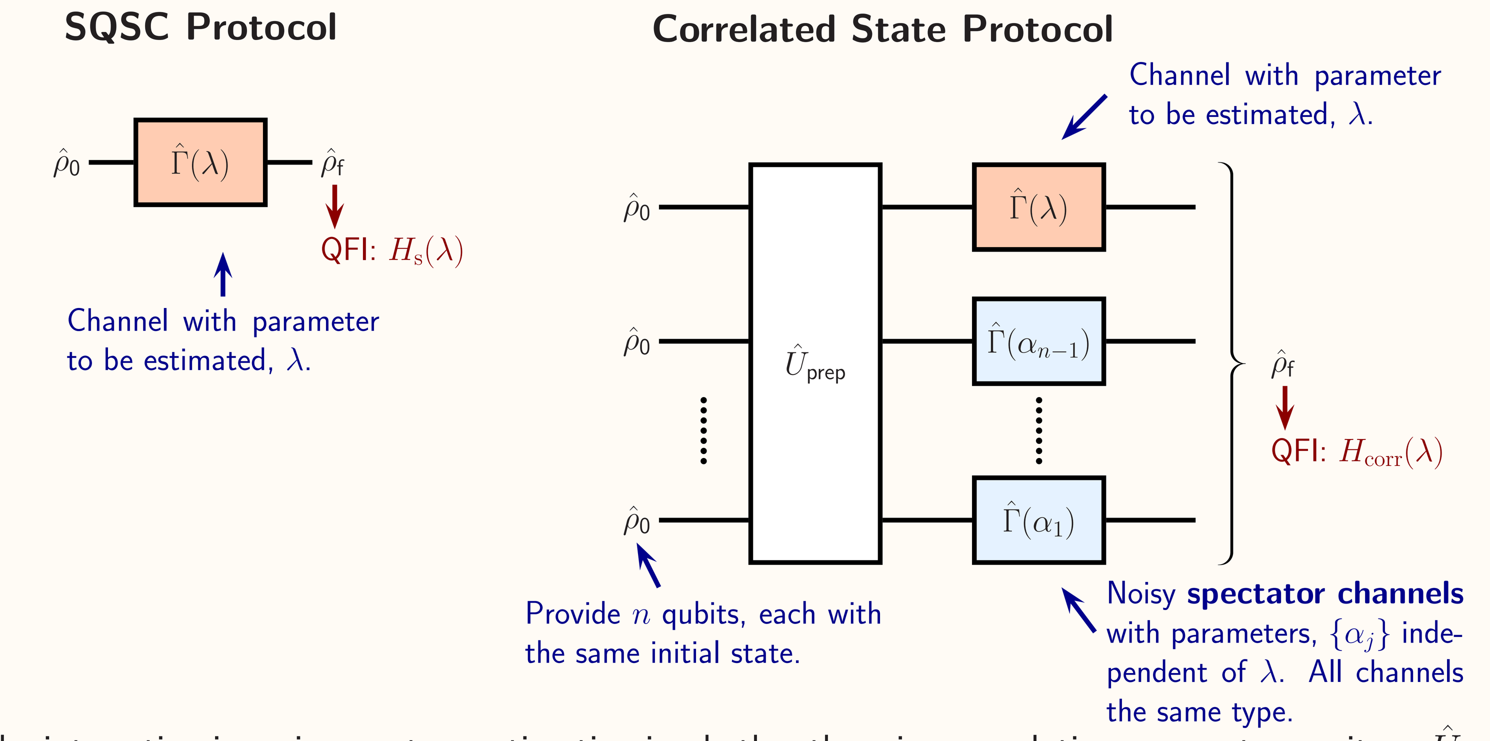
$$H(\lambda) = \text{Tr}\left(\hat{\rho}_{\text{f}} \hat{L}^2\right) \quad \text{with} \quad \frac{\partial \hat{\rho}_{\text{f}}}{\partial \lambda} = \frac{1}{2}\left(\hat{\rho}_{\text{f}} \hat{L} + \hat{L} \hat{\rho}_{\text{f}}\right)$$

where $\hat{\rho}_{\text{f}}(\lambda)$ is the state of the system after evolution and before measurement.
The task is to find input states that maximize the QFI for given types of evolutions; these are always attained using pure input states [2]. For single systems in pure input states, additional qubits and entanglement can enhance the QFI and thus improve estimation accuracy [2]. However, for some quantum channels no such enhancement exists beyond a single qubit or a pair of qubits when they are in a pure initial state [3, 4]. These results assume that pure input states are available.

Consider situations when **optimal pure states are not available.** For systems **restricted to highly mixed initial states, can the QFI be increased by involving spectators?**

## Competing Estimation Protocols

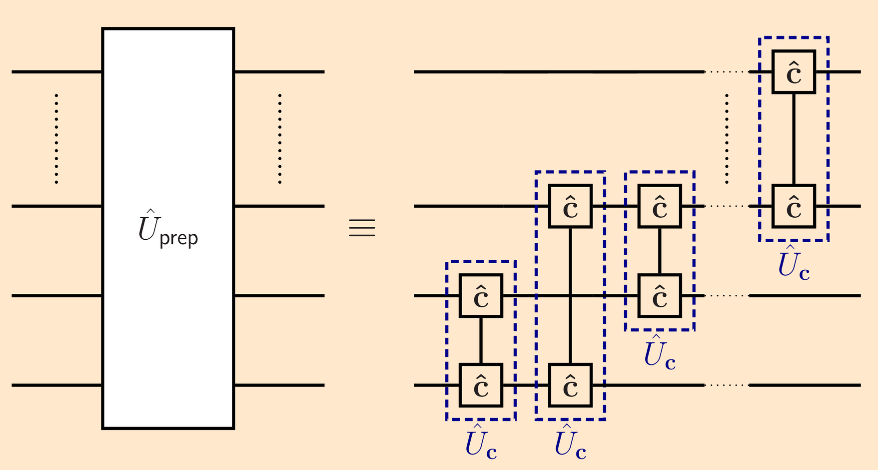
Consider competing protocols that each use one invocation of the channel with the parameter $\lambda$ to be estimated: a single-qubit, single channel (SQSC) protocol and a correlated state protocol.




The interesting issue in quantum estimation is whether there is a correlating preparatory unitary, $\hat{U}_{\text{prep}}$, such that the correlated state protocol QFI, $H_{\text{corr}}(\lambda)$, exceeds the SQSC protocol QFI, $H_{\text{s}}(\lambda)$. This is often resolved by identifying a particular preparatory unitary.

## Symmetric Pairwise Correlated Protocol

A particular preparatory unitary using one correlating gates on each pair of qubits [8] generalizes the scheme of [5, 6, 7]. This is the correlated state protocol discussed henceforth.




$$\hat{U}_{\hat{\mathbf{c}}} := \frac{1}{2}\left(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_c + \hat{\sigma}_c\otimes\hat{I} - \hat{\sigma}_c\otimes\hat{\sigma}_c\right)$$

where $\hat{\sigma}_c = \hat{\mathbf{c}}\cdot\hat{\boldsymbol{\sigma}}$ where $\hat{\mathbf{c}}$ is a unit vector (Bloch sphere control direction) chosen depending on the channel type.

## Pure vs. Noisy Initial States

The general qubit initial state is

$$\hat{\rho}_0 = \frac{1}{2}(\hat{I} + r\hat{\mathbf{r}}_0\cdot\hat{\boldsymbol{\sigma}})$$

where $0 \leqslant r \leqslant 1$ is the **purity** and $\hat{\mathbf{r}}_0$ is a unit vector (initial Bloch sphere direction). We compare situations where all qubits have the same initial purity and subsequent evolutions are either unitary or the channels containing the parameter or else channels of the same type on the spectators. We ask whether a symmetric pairwise protocol enhance estimation accuracy versus the SQSC protocol where **both utilize the same initial state purity.**
The answer depends on the channel and the initial state purity [3, 2, 4, 6, 7]. **If there is no noisy evolution on the spectator qubits** then for very low purity ($r \to 0$):

| Channel | Pure Initial State: QFI Enhancement? | Very Low Purity Initial State: QFI Enhancement? |
|---|---|---|
| **Phase-shift** | No | Yes, by factor of $n$ |
| **Phase-flip** | No | Yes, by factor of $n$ |
| **Depolarizing** | Only for $n=2$ | Yes, by factor of $n$ |

For very low purity the **correlated state protocol enhances estimation accuracy for unital channels by a factor of at least** $n-1$ when there is no post-preparation noise on the spectators [8].

## Low Purity Initial States with Noisy Spectators

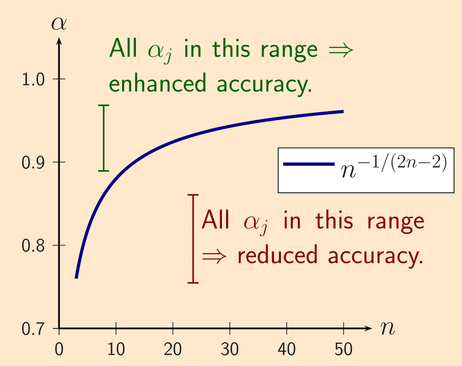
Noisy evolution may also be present on the spectators after preparation.

**How will additional post-preparation noise on the spectators affect accuracy enhancement?**

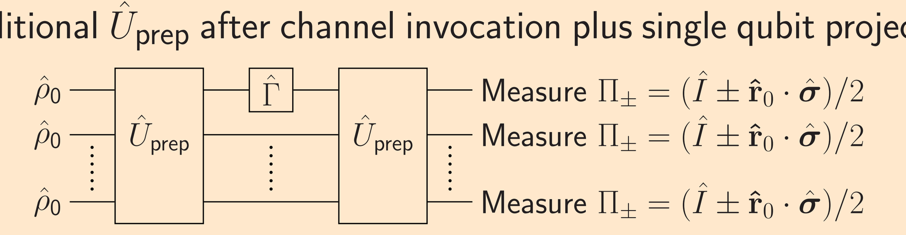
| Primary Channel | Spectator Channels | SQSC QFI | Correlated QFI |
|---|---|---|---|
| **Phase-shift:** $\hat{\rho}_{\text{f}} = e^{-i\lambda\hat{\sigma}_z/2}\hat{\rho}_{\text{i}}e^{i\lambda\hat{\sigma}_z/2}$ | Phase-shift | $r^2$ | $\boldsymbol{n}\, r^2$ |
| **Phase-flip:** $\hat{\rho}_{\text{f}} = (1-\lambda)\,\hat{\rho}_{\text{i}} + \lambda\hat{\sigma}_z\hat{\rho}_{\text{i}}\hat{\sigma}_z$ | Phase-flip $\hat{\rho}_{\text{f}} = (1-\alpha)\,\hat{\rho}_{\text{i}} + \alpha\hat{\sigma}_z\hat{\rho}_{\text{i}}\hat{\sigma}_z$ | $4r^2$ | $\boldsymbol{n}\,(\mathbf{1}-\mathbf{2}\boldsymbol{\alpha}_{\boldsymbol{n-1}})^{\mathbf{2}}\cdots(\mathbf{1}-\mathbf{2}\boldsymbol{\alpha}_{\mathbf{1}})^{\mathbf{2}}\, 4r^2$ |
| **Depolarizing:** $\hat{\rho}_{\text{f}} = \frac{1-\lambda}{2}\hat{I} + \lambda\hat{\rho}_{\text{i}}$ | Depolarizing $\hat{\rho}_{\text{f}} = \frac{1-\alpha}{2}\hat{I} + \alpha\hat{\rho}_{\text{i}}$ | $r^2$ | $\boldsymbol{n}\,\boldsymbol{\alpha}^{\mathbf{2}}_{\boldsymbol{n-1}}\cdots\boldsymbol{\alpha}^{\mathbf{2}}_{\mathbf{1}}\, r^2$ |

**Enhancement is possible** with spectator qubits but is offset by the additional noisy post-preparation spectator evolution. **For noisy spectator qubits, beneath a certain number of qubits, the correlated protocol will enhance accuracy. Above this it will reduce accuracy.**

**Example:** If the channels are all depolarizing then the possibility of definite accuracy enhancement or definite reduction depends on whether $\alpha_1, \ldots, \alpha_{n-1}$ all lie above or below $n^{-1/(2n-2)}$. If their range straddles this division then enhancement depends on their relative values.




It remains to identify a measurement scheme such that the resulting classical Fisher information attains the QFI. For the correlated state protocol such a measurement scheme does exist and for the listed channels **there is a local, parameter-independent saturating measurement.** This uses one additional $\hat{U}_{\text{prep}}$ after channel invocation plus single qubit projective measurements.




The analysis is valid when $nr^2 \ll 1$ or when $n \ll 1/r^2$. For example, for liquid-state NMR $r \approx 10^{-4}$ and this analysis would be applicable if $n \ll 10^8$.

For pure input states, additional spectators only enhance estimation accuracy in special cases. For **very noisy input states, estimation accuracy is more generally enhanced by a factor of at least the number of spectators although this is offset by additional noise on the spectators.**

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