Qubit Channel Parameter Estimation with Very Noisy Initial States

David Collins

Department of Physical and Environmental Sciences, Colorado Mesa University, Grand Junction, CO 81501, USA

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Abstract

The accuracy of physical processes for estimating parameters associated with single qubit channels depends on the physical systems used to probe the channel, the choices of measurements, processing of measurement outcomes and the choices of probe input states. These can be assessed using the quantum Fisher information per channel invocation as a measure of the estimation accuracy. The resulting optimal estimation protocols usually require the initial states that are used to generate the input states to be pure.

We consider qubit channel parameter estimation when the available initial states are mixed with very low initial purity, r. We compare two protocols: one where the input states into the channel are uncorrelated states generated independently from the individual qubit initial states and the other where the input states are prepared from the same initial states using a particular multi-qubit correlating preparatory unitary. We compare these for the cases where the channel is invoked on one out of n qubits and whenever the purity satisfies $r << 1/\sqrt{n}$. We show that for unital channels the correlated state protocol enhances the quantum Fisher information by a factor between n and $n-1$. We also show that for a broad class of non-unital channels, there is no enhancement possible to lowest order in purity, regardless of the input state.

Quantum Channel Parameter Estimation

Channel: known form, unknown parameter, λ.

Input state $\rightarrow$ $\hat{\rho}_{\text{i}}$ —$\boxed{\hat{\Gamma}(\lambda)}$— $\hat{\rho}_{\text{f}}(\lambda)$ $\leftarrow$ Parameter dependent output state.

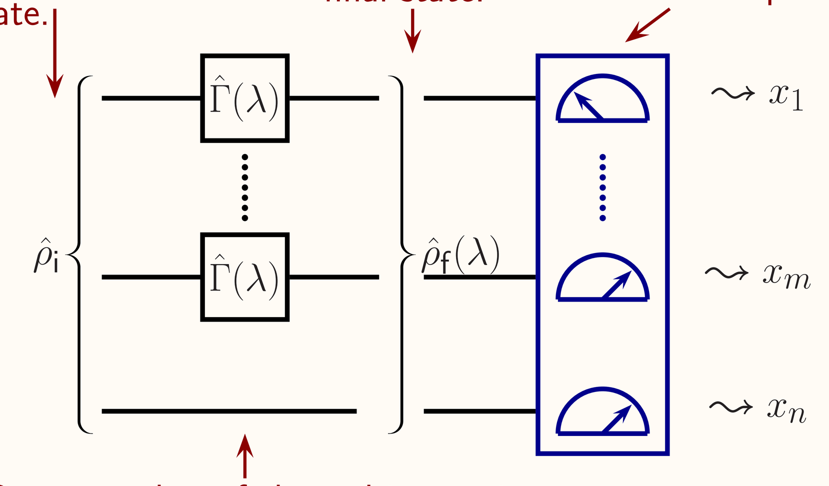
Task: Estimate parameter by subjecting quantum system to channel and inferring λ from measurements on output state.

Quantum Fisher Information

Choose n qubit input state.

Post-evolution final state.

Choose joint measurement on all qubits. List outcomes.



Choose estimator function. Gives estimate:

$$\lambda_{\text{est}} = \lambda_{\text{est}}(x_1, \ldots, x_n)$$

Goal: Unbiased estimate; smallest fluctuations.

Cost: number of channel invocations, m.

For any unbiased estimator, the Cramér-Rao bound [1] bounds the mean square error:

MSE depends on estimator choice.

$$\text{m.s.e}\left(\lambda_{\text{est}}\right) = \left\langle \left(\lambda_{\text{est}} - \lambda\right)^2 \right\rangle \geqslant \frac{1}{F(\lambda)} \geqslant \frac{1}{H(\lambda)}.$$

Classical Fisher information depends on measurement choice, not estimator.

QFI depends on $\hat{\rho}_{\text{f}}$ but not measurement choice nor estimator.

The quantum Fisher information (QFI) is given by

$$H(\lambda) = \text{Tr}\left(\hat{\rho}_{\text{f}} \hat{L}^2\right) \quad \text{with} \quad \frac{\partial \hat{\rho}_{\text{f}}}{\partial \lambda} = \frac{1}{2}\left(\hat{\rho}_{\text{f}} \hat{L} + \hat{L}\hat{\rho}_{\text{f}}\right).$$

For multiple channel invocations choose number of qubits, n, and their input state, $\hat{\rho}_{\text{i}}$, to **maximize QFI per channel invocation.**

Pure input states are optimal.

Absolute optimal QFI per channel invocation is attained using pure input states [2]. Product input states cannot enhance the QFI per channel invocation but sometimes entangled pure input states can. For qubit channels:

Channel	**Entangled pure input states QFI enhancement**?
Phase-shift	Yes, by factor of number of channel invocations, n [3].
Phase-flip	Impossible [4].
Depolarizing	Only with pairs of entangled qubits [2].

Pure input states may not be available.

In situations such as room-temperature liquid-state NMR, the initial states of qubits are mixed and pure input states cannot be prepared from these. Assume the initial state for multiple qubits is $\hat{\rho}_0 \otimes \hat{\rho}_0 \otimes \cdots \otimes \hat{\rho}_0$ where

$$\hat{\rho}_0 = (\hat{I} + r\hat{\mathbf{r}}_0 \cdot \hat{\boldsymbol{\sigma}})/2$$

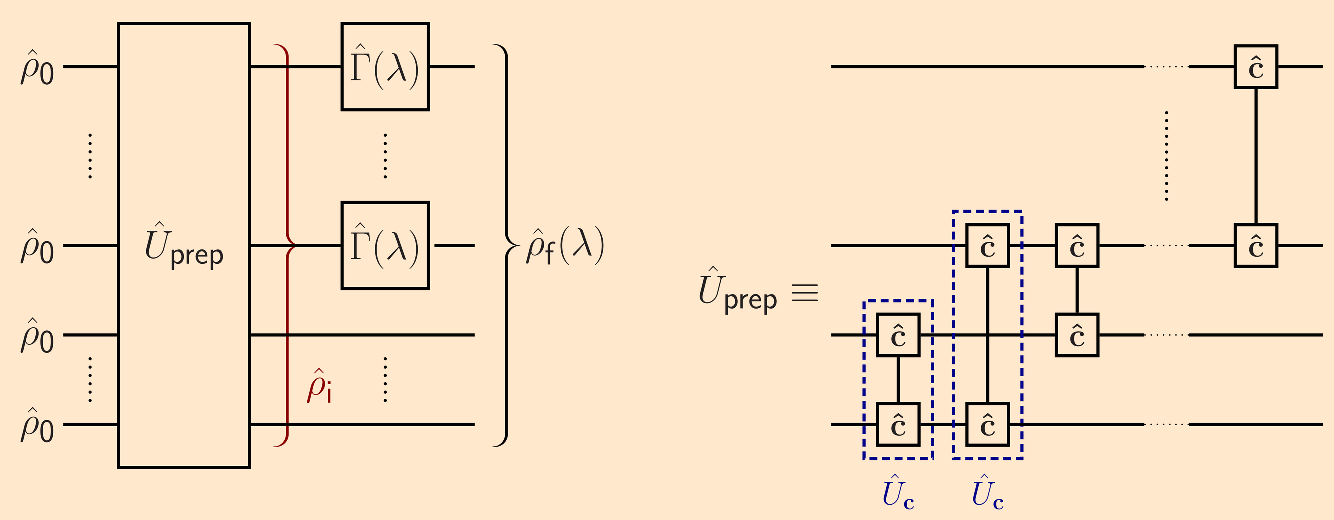
where r is the **purity** and $\hat{\mathbf{r}}_0$ an initial Bloch sphere direction. Any input state will be created by invoking a multi-qubit preparation unitary, $\hat{U}_{\text{prep}}$ giving $\hat{\rho}_{\text{i}} = \hat{U}_{\text{prep}}\, \hat{\rho}_0 \otimes \hat{\rho}_0 \otimes \cdots \otimes \hat{\rho}_0\, \hat{U}_{\text{prep}}^\dagger$.

Independent/product-state protocol: $\hat{U}_{\text{prep}}$ is a product of single qubit unitaries.
Correlated-state protocol: $\hat{U}_{\text{prep}}$ is not product of single qubit unitaries.

Compare protocols with mixed initial states with the same initial state purity. Could correlated input states enhance QFI per channel per channel invocation over independent-state protocols?

Symmetric Pairwise Correlated Protocols

Particular correlating preparatory unitary generates input state.



This is a generalization of the scheme of [5, 6] with

$$\hat{U}_{\hat{\mathbf{c}}} := \frac{1}{2}\left(\hat{I} \otimes \hat{I} + \hat{I} \otimes \hat{\sigma}_c + \hat{\sigma}_c \otimes \hat{I} - \hat{\sigma}_c \otimes \hat{\sigma}_c\right)$$

where $\hat{\sigma}_c = \hat{\mathbf{c}} \cdot \hat{\boldsymbol{\sigma}}$ where $\hat{\mathbf{c}}$ is a unit vector (Bloch sphere control direction).

Previous results for the phase-shift [5], phase-flip [6] and depolarizing channels [7] showed that with m channel invocations on n qubits, initially restricted to mixed initial states:

- For some purities, parameter values and number of channel invocations, a correlated state protocol will enhance the QFI versus the independent state protocol with the same purity.
- If the channel is invoked once on a single qubit (out of n total), then as the purity $r \rightarrow 0$, the correlated-state protocol gives an n-fold enhancement of the QFI.

What accuracy enhancements does the correlated state protocol provide for general single qubit channels as the initial state purity $r \rightarrow 0$?

Estimation with Low Purity

For general channels, closed form expressions for the QFI are difficult to obtain. Approximate expressions for the QFI can be obtained by expressing $\hat{\rho}_{\text{f}}, \hat{L}$ and H as sequences in increasing powers of r and then solving for the QFI order-by order.

Compare the independent protocol to correlated protocol using a channel invocation on one out of n qubits in the limit as $r \rightarrow 0$.

Single qubit channel:

$$\hat{\rho}_{\text{i}} = \frac{1}{2}\left(\hat{I} + r\hat{\mathbf{r}}_{\text{i}} \cdot \hat{\boldsymbol{\sigma}}\right) \mapsto \frac{1}{2}\left(\hat{I} + \mathbf{r}_{\text{f}} \cdot \hat{\boldsymbol{\sigma}}\right)$$

Channel described by Bloch-sphere map:

$$\mathbf{r}_{\text{f}} = rM(\lambda)\hat{\mathbf{r}}_{\text{i}} + \mathbf{d}(\lambda)$$

$\rightarrow$ Singular values of $\dot{M} = \frac{\partial M}{\partial \lambda}$:

$$s_1 \geqslant s_2 \geqslant s_3 \geqslant 0$$

Non-unital channel with $\dot{\mathbf{d}} \neq 0$:

Optimal independent protocol QFI:

$$H_{\text{s opt}}(\lambda) = \begin{cases} \dot{\mathbf{d}} \cdot \dot{\mathbf{d}} + \dfrac{1}{4(1-d^2)}\left[\dfrac{\partial d^2}{\partial \lambda}\right]^2 & \text{if } d \neq 1 \\ \dot{\mathbf{d}} \cdot \dot{\mathbf{d}} & \text{if } d = 1. \end{cases}$$

Optimal correlated protocol QFI:

$$H_{\text{corr/channel}}(\lambda) = H_{\text{s}}(\lambda)$$

Correlated protocol offers no gain.

Unital channel ($\mathbf{d} = 0$):

Optimal independent protocol QFI:

$$H_{\text{s}}(\lambda) = r^2 s_1^2$$

Optimal correlated protocol QFI :

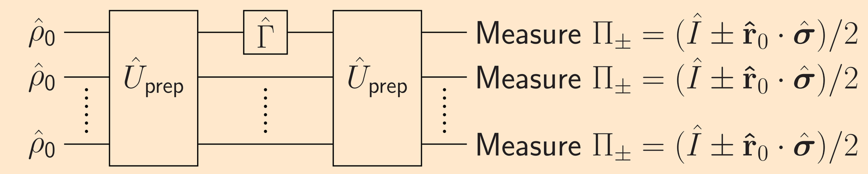
$$n - \left(1 - \frac{s_2^2}{s_1^2}\right) \leqslant \frac{H_{\text{corr}}(\lambda)}{H_{\text{s}}(\lambda)} \leqslant n$$

Correlated protocol offers gain by a factor of at least $n-1$.

For a single unital channel acting on one out of n qubits, the optimal symmetric-pairwise correlated protocol has lowest order term $r^2 n$. The next non-zero term is of order $(r^2 n)^2$. The analysis applies if $n \ll 1/r^2$. For liquid-state NMR $r \approx 10^{-4}$ and this analysis would be applicable if $n \ll 10^8$.

Is there a local measurement scheme for attaining enhanced QFI?

One additional $\hat{U}_{\text{prep}}$ after channel invocation plus single qubit projective measurements.



Particular choices of, $\hat{\mathbf{r}}_0$ and $\mathbf{c}$, determined from singular value decomposition of $\dot{M}$, give classical Fisher information, F, such that $F/H_{\text{S}}(\lambda) = n - \left(1 - s_2^2/s_1^2\right)$, saturating lower bound and giving a **gain factor of at least** $n-1$.

For any unital channel, the symmetric pairwise correlated protocol using one channel invocation on a single qubit out of n always yields an approximately n-fold enhancement in QFI versus any independent channel protocol, provided $r \ll 1/\sqrt{n}$. Such gains do not always arise when available inputs states are pure.

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