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George Siopsis

Major Professor

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Henry Simpson

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Anne Mayhew

Vice Chancellor and
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Gravity in Extra Dimensions of Infinite Volume

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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Chad Aaron Middleton

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To my parents, Don and Lana...

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Abstract

In this thesis, we discuss various aspects of the Dvali-Gabadadze-Porrati (DGP) model in D-dimensions. Firstly, we generalize the DGP model, which consists of a delta-function type 3-brane embedded in an infinite volume bulk-space by allowing the 3-brane to have a finite thickness. We calculate the graviton propagator in the harmonic gauge both inside and outside the brane and discuss its dependence on the thickness of the brane. We obtain two infinite towers of massive modes and tachyonic ghosts. In the thin-brane limit, we recover the four-dimensional Einstein gravity behavior of the graviton propagator which was found in the delta-function treatment. We then examine the 4D worldvolume momentum dependence of the tensor structure. Secondly, we address the van Dam-Veltman-Zakharov (vDVZ) discontinuity of the 5D DGP model which arises from the breakdown of the perturbative expansion at linear order. Following a suggestion by Gabadadze `[hep-th/0403161]`, we implement a constrained perturbative expansion parametrized by brane gauge-like parameters. We obtain the solution for the metric perturbations, explore the parameter space and show that the DGP solution exhibiting the vDVZ discontinuity corresponds to a set of measure zero. Thirdly, we discuss the weak-field Schwarzschild solution in the DGP model. By keeping up to second-order off-diagonal terms of the metric ansatz, we arrive at a perturbative expansion which is valid both far from and near the Schwarzschild radius. We calculate the lowest-order contribution explicitly and obtain the form of the metric both on the brane and in the bulk. As we approach the Schwarzschild radius, the perturbative expansion yields the standard four-dimensional Schwarzschild solution on the brane which is non-singular in the decoupling limit. This non-singular behavior is similar to the Vainshtein solution in massive gravity demonstrating the absence of the vDVZ discontinuity of the DGP model.

Contents

Chapter 1

Introduction and Summary

Dimensionality has long been considered by mankind since antiquity. Perhaps Pythagoras of Samos (569-500 B.C.) should be accredited with the earliest mathematical treatment of the concept of dimensionality when he provided the proof which relates the three sides of a right triangle through the famous equation which bears his name. Aristotle (384-322 B.C.) a century later pondered the concept of dimensions and in his work *On Heaven* wrote “The line has magnitude in one way, the plane in two ways, and the solid in three ways, and beyond these there is no other magnitude because the three are all”. This declaration of space being strictly limited to three spatial dimensions was taken further when Ptolemy (85-165 A.D.), the last great Alexandrian astronomer, suggested a ‘proof’ of the non-existence of extra dimensions in his work *On Distance*. Even Euclid (325-265 B.C.), the most prominent mathematician of ancient Greece, neglected to even consider the possibility of higher dimensions in his best known treatise on mathematics *The Elements*, which has been the center of mathematical teaching for 2000 years and comprises what is now known as Euclidean geometry.

On June 10, 1854, Bernhard Riemann presented a lecture on his habilitation, the degree which would allow him to become a lecturer, entitled *On the hypotheses* *that lie at the foundation of geometry.* This lecture presented a new way of looking at geometry which generalized Euclidean geometry to non-Euclidean geometry allowing for any number of higher dimensions and for curved surfaces. Although Riemann himself did not give much thought to the physical reality of extra dimensions, his ideas planted the seed for serious inquiry into a realization of higher dimensions.

In 1905, Albert Einstein published his work on special relativity which shattered the long standing Newtonian concepts of absolute time and space. The century prior had seen the unification of electricity and magnetism via Maxwell's equations which predicted the propagation of electromagnetic waves (one example being that of visible light) traveling at a constant speed c for all inertial reference frames. Using this in addition to the postulate that there exists only relative motion, Einstein constructed this theory making use of the Lorentz transformations which relate two inertial coordinate systems moving with different relative speeds. These transformations effectively mix measurements of location and time and thus paved the way for treating space and time on equal footings. Hermann Minkowski furthered the marriage of space and time into the concept of spacetime in 1909 by giving a geometrical interpretation of special relativity. He realized that if he treated time as an imaginary coordinate, then the Lorentz transformations can be thought of as rotations in this four-dimensional spacetime. After Minkowski, Einstein finally completed the spacetime merger in 1915 when he introduced his General Theory of Relativity, a generally covariant four-dimensional theory of gravity. Using Riemann's geometric formalism of higher dimensional spaces, Einstein constructed the field equations describing gravity as the curvature of spacetime. This, consequently, brought Riemann's theory of higher dimensions and curved surfaces out of the academic realm of pure mathematics and gave it a physical realization. This general theory, which assumes complete equivalence of a gravitational field and the corresponding acceleration of the reference frame, unifies the special theory of relativity with Newtonian gravitation.

In the years shortly after the onset of Einstein's general theory of relativity, physics embodied two different field theories describing the two known fundamental forces of Nature, electromagnetism and gravitation. Using the mathematical framework of general relativity, Theodor Kaluza and Oskar Klein attempted to unify the two forces into a more fundamental description. In 1919, Kaluza [1] showed that by generalizing the Einstein field equations in vacua to five dimensions and by making an appropriate metric *ansatz*, one could recover both the 4D Einstein field equations describing gravity and Maxwell's equations of electricity and magnetism. Besides the obvious problem of having to hypothesize an undetected spatial dimension, Kaluza had to assume that all metric components were independent of this extra dimension. Expanding on the original work of Kaluza, Klein [2] provided a resolution to the conflict of the metric's independence of the extra spatial dimension. Klein assumed that the extra dimension was compactified in such a way that at every point in 4D spacetime, there exists a small circle on the order of the Planck length. This allows for a Fourier expansion of the periodic extra dimension and yields a tower of Kaluza-Klein (KK) massive modes. The energy levels of these massive modes are inversely proportional to the compactification radius, thus, probing small compactified extra dimensions requires huge amounts of energy which are not accessible in the conventional low energy experiments (for compactification radii on the order of the Planck scale). In the low energy regime, and thus a large distance regime, physics of the KK theory appears four-dimensional. It is not until small distances are probed, requiring large energies, that physics becomes that of a five-dimensional theory. Conceptually, the example of a garden hose is often used for illustrative purposes. From a distance, a garden hose appears as a one-dimensional object and not until closer examination is the two-dimensional surface revealed.

Although not accepted as a correct fundamental theory to describe Nature, the KK mechanism of adding extra dimensions has been actively studied during the last several years. The advent of string theory as a possible candidate to offer a consistent quantum description of gravity has been primarily responsible for fueling this activity in the research of extra dimensional scenarios as superstring theory has the strict requirement of residing in a 10D spacetime. Traditionally, the six extra spatial dimensions predicted by string theory have been treated as compactified objects, as are the dimensions of the Kaluza-Klein theory. The discovery of D(irichlet)-branes as fundamental extended objects has provided a new possible discription of extra spatial dimensions and has given rise to several different braneworld scenarios where the extra dimensions are large. These braneworld scenarios describe a three dimensional dynamical hypersurface, a 3-brane for the sake of brevity, embedded in a higher dimensional bulk. The open strings, which are the Standard Model (SM) particles, are confined to the brane, however, the closed strings corresponding to spin-2 gravitons have no such boundary requirements and are free to propagate on the brane and in the bulk.

In order to obtain a realistic braneworld model which agrees with astronomical data, one must obtain 4D gravity on the brane. The Newtonian distance dependence of gravitational interactions has been well tested from the submillimeter (.2mm) regime up to the cosmological horizon (10^{26}cm), which corresponds to about 1% of the size of the observable universe [56]. Any deviation from a Newtonian potential is strictly prohibited within this distance regime, however, beyond the cosmological horizon there is nothing constraining an alternative theory from predicting a departure from that of the 4D general relativity.

Currently, there are three known mechanisms for obtaining the 4D laws of gravity on a brane residing in large extra dimensions. These different braneworld scenarios residing in a higher dimensional bulk have successfully explained the weakness of the gravitational force. The first is to combine the braneworld idea with a KK compactification of $(D-4)$-dimensions with large compactification radius. This scenario was proposed in 1998 by Arkani-Hamed, Dimopoulos, and Dvali (ADD) [3, 4, 5] and offers a way to eliminate the Higgs mass hierarchy problem as the fundamental plank mass scale is lowered to that of the order of the weak scale. A second known way was discovered by Randall and Sundrum [6, 7] in 1999. The RSII model [7] involves a single brane embedded in a 5 dimensional bulk-space with a negative cosmological constant and non-vanishing brane tension. By fine-tuning the values of the bulk cosmological constant *and* the brane tension, a solution for the metric can be obtained which exhibits a *warped* bulk-space. Although the bulk coordinate is not compactified and runs in an infinite interval, the physical size of the extra dimension is finite and the warp factor provides for a localization of gravity. The effect of these scenarios is a high-energy modification of Newton's Law of gravity due to the tower of Kaluza-Klein modes.

In this thesis, we will discuss the third scenario being that of Brane Induced Gravity, in particular, the model proposed in 2000 by Dvali, Gabadadze, and Porrati (DGP) [8, 9] which describes a 3-brane residing in an infinite-volume extra space. When the extra dimensions are of infinite volume, light Kaluza-Klein modes may dominate even at low energies [8, 9, 10], therefore offering an attractive alternative to dark energy for solving the cosmological constant problem [11, 12]. Thus, unlike with finite-volume extra space, Newton's Law is modified at astronomically large distances [13, 14, 15, 16, 17, 18, 19, 20, 21].[1]

Dvali and Gabadadze [9] showed that this is not the case if the infinite space in which the brane lives has dimension $D > 5$. They studied a three-brane of the δ-function type and showed that the graviton propagator has a four-dimensional momentum dependence on the brane even at low energies. This feature is not expected to persist if the brane is of finite thickness ("fat") in the transverse directions for phe-

[1] See [44] for a slightly different treatment which yields a $4D$ tensor structure at astronomical distances.

nomenologically relevant values of the momentum. This was discussed qualitatively in [9]. For low energies (large distances), the fat brane treatment should lead to a higher-dimensional behavior of the graviton propagator [22, 23, 24].

The DGP model of a 3-brane residing in a $5D$ bulk of vanishing cosmological constant [8] is a ghost-free, general covariant theory where the $5D$ graviton mimics a $4D$ massive graviton on the brane. The model appears to be plagued by a van Dam-Veltman-Zakharov (vDVZ) discontinuity [30, 31], as does $4D$ Pauli-Fierz massive gravity [45] at linear order, where one does not obtain agreement with Einstein's General Theory of Relativity in the vanishing mass limit of the graviton, and has attracted much attention [33, 34, 35, 36, 37, 38, 39, 40, 41]. Vainstein [32] provided a solution to the apparent discontinuity for the $4D$ Pauli-Fierz model in the case of a point source by suggesting that the discrepancy arises from the linear approximation to the full field equations which has a limited range of validity. This second solution reduced to the Schwarzschild solution in the zero graviton mass limit demonstrating the absence of the vDVZ discontinuity in this spherically symmetric case. [2] Applying a similar procedure to the DGP model is not straightforward, because the non-linear field equations are too complicated to solve even in the spherically symmetric case of a point source. In refs.[37], solutions for the DGP model were found interpolating between regimes far from and near the Schwarzschild radius by keeping higher-order terms in the perturbative expansion. It was thus shown that in the decoupling limit, one recovers the standard four-dimensional, weak-field Schwarzschild metric.

As has been recently argued in [46, 47] for the specific case of $D = 5$, the breakdown of the perturbative expansion at linear order is an artifact of the weak-field expansion itself and can be healed by adopting a *constrained* perturbative expansion. Thus, instead of the incorporation of higher-order terms into the linearized treatment, the theory is regulated by a modification of the linearized theory itself. After fixing

[2]See [43] for problems associated with Vainstein's approach.

the gauge in the bulk, a residual four-dimensional gauge invariance remains on the brane. The graviton propagator is then rendered invertible by the addition of a term in the action which amounts to a gauge-fixing term in four-dimensional gravity.

This dissertation is organized as follows.

In Chapter 2, we begin by introducing the salient generic features of Brane Induced Gravity in an infinite-volume bulk-space. By limiting our analysis to a simplest setup of keeping only lowest-order derivative terms for a Minkowski bulk and fine tuning the brane cosmological constant to exactly cancel the brane tension, we arive at the model of Dvali-Gabadadze-Porrati (DGP). The initial work in [8, 9] provides much of the motivation for the research presented in this thesis and will thus be thoroughly re-examined here. In section 2.1, we review the original treatment of the DGP model and arrive at the solution for the graviton propagator in $D = 5$ [8] and $D > 5$ [9] for a delta-function type brane. For the case of $D > 5$, the solution for the graviton propagator was found to have a 4D tensor structure and distance dependence on the brane. This solution was found in a singular manner, as the Green function is discontinuous at the location of the brane, and suggests that a regularizational scheme should be adopted. To regulate the theory, one can either keep higher-dimensional derivative terms obtained from next order contributions to the bulk, which in the simplest setup are neglected, or alternatively one can allow for a brane of finite thickness (making the brane "fat"). This finite thickness brane can arise if the brane is treated as a smooth soliton in the bulk or by transverse fluctuations of the brane into the bulk-space giving the brane an effective thickness. In the Chapter 3, we regulate the theory by choosing the latter. For the case of $D = 5$, the solution for the metric perturbations was found to coincide with that of tensor-scalar gravity. This solution has a 4D distance dependence in the near regime crossing over to a 5D distance dependence in the far regime. However, when one examines the full tensor structure, one finds that the solution for the metric perturbations appear to suffer from a van Dam-Veltman-Zakharov (vDVZ) discontinuity similar to that of the Pauli-Fierz model of massive gravity at linear order. In the decoupling limit, the solution for the metric perturbations does not reduce to that of a 4D behavior signalling disagreement with known astronomical data. This discontinuity is due to the breakdown of the perturbative expansion due to the weak field.

In Chapter 3, a brane-bulk action is considered which is similar to the action of the DGP model for the delta-function type [9] but generalized to allow for a brane of finite thickness with extent into the infinite volume bulk-space. By linearizing gravity in the harmonic gauge, we arrive at an explicit expression for the graviton propagator in section 3.2. First, we obtain the propagator for the trace of the metric field over the transverse directions. The trace is a scalar field from the four-dimensional brane point of view. It is then found that this scalar contributes to the four-dimensional graviton propagator as a source, in addition to the matter fields. This complicates the tensor structure of the graviton propagator which becomes momentum dependent. We explicitly obtain the solution for the graviton propagator and proceed by analyzing it's pole structure and momentum dependence. In section 3.3, we find two infinite towers of massive modes and tachyonic ghosts. It is found that in the thin-brane limit, the infinite towers merge into a continuous spectra. The terms that give rise to the poles become vanishingly small and we recover four-dimensional Einstein gravity on the brane; the solution for the graviton propagator reduces to that of the delta-function type setup. It is found that the pole corresponding to the massless propagator of Einstein gravity is independent of the bulk coordinates on the brane. In section 3.4, the momentum dependence of the propagator is then analyzed and we demonstate how the graviton propagator changes from one of 4D behavior in both the tensor structure and distance dependence to that of D dimensional behavior.

In Chapter 4, we address the vDVZ discontinuity of 4D massive gravity which arises in the DGP model for the unique case of $D = 5$ dimensions. In section 4.1, we rederive the Pauli-Fierz model of massive gravity in the context of the treatment of the latter part of this chapter which is described as follows. We add to the generally covariant 4D Einstein-Hilbert action an additional linear action contribution. This additional action contribution is left completely general and is written in terms of free parameters which consequently exhaust all possible combinations of the metric perturbations which yield local field contributions. By solving the linearized field equations and demanding the theory be free of tachyonic and ghost-like states, the values of the free parameters become severely constrained and hence, we arrive at the Pauli-Fierz model of massive gravity [42]. As has been recently argued in [46, 47] for the 5D DGP model, the breakdown of the perturbative expansion at linear order is an artifact of the weak-field expansion itself and can be healed by adopting a *constrained* perturbative expansion. In section 4.2, we present a generalized procedure of [46] with the intent of regulating the linearized theory itself without the inclusion of higher order field contributions. As was done in the previous section, we add an additional worldvolume action contribution to the model which is written in terms of free brane parameters. Accompanying this new linear brane term, we also include an additional bulk contribution also written in terms of free parameters. The inclusion of these additional terms explicitly break the bulk and brane gauge invariance. In the decoupling limit and in the absence of the brane, these additional terms reduce to mere gauge-fixing conditions. Away from these limits, the additional terms regulate the theory and modify the linear field equations. After expanding around a Minkowski background and obtaining the field equations, we obtain the solution for the metric perturbations. We find a set of constraint equations which limits the domain of the bulk parameters. We find that the solution has the expected 4D momentum-dependent crossover behavior and is independent of the brane and bulk parameters in the small and large 4D momentum regimes. We then examine the pole structure by exploring the parameter space and identify the region which yields non-tachyonic type resonances.

In Chapter 5, we readdress the vDVZ discontinuity of the DGP model in the context of a massive point source. The vDVZ discontinuity of the Pauli-Fierz model was addressed by Vainstein [32] who provided a solution to the apparent discontinuity, in the case of a massive point source, by suggesting that the discrepancy arises from the linear approximation to the full field equations which have a limited range of validity. In section 5.1, we review the salient features of [32]. By choosing a spherically symmetric metric ansatz, expanding the field equations in the small graviton mass regime and keeping up to second-order terms, we obtain the Vainstain solution. This solution is well-behaved in the vanishing graviton mass limit and corresponds to that of 4D massless gravity . In section 5.2, we follow a similar approach for the 5D DGP model for the case of a massive point source. We choose a spherically symmetric metric ansatz with the inclusion of an off-diagonal metric contribution and obtain the DGP field equations. We expand around a flat background keeping all lowest order field contributions plus second-order contributions in the off-diagonal metric term. We obtain a set of coupled non-linear field equations which can be decoupled and solved and find that the solution is valid in the near and far regime. In the far regime, the solution is that which emerges from the linear perturbative expansion . In the near regime, the second-order contribution of the off-diagonal metric field yields a non-negligible contribution and we obtain a solution which smoothly transitions to the 4D Einstein solution thus showing an absence of the vDVZ discontinuity when the correct expansion is performed.

Finally, in Chapter 6 we conclude the thesis. We summarize our main findings and discuss some of the attractive features of the DGP model.

Chapter 2

Brane Induced Gravity

We begin by constructing the general form of a bulk action describing a non-compactified infinite-volume, D-dimensional extra space which is asymptotically flat at infinity [8, 9]. Suppose a general D-dimensional bulk-space action of the form

$$S_{\text{bulk}} = \int d^D X \sqrt{-G}\, \mathcal{L}(G_{AB}, \mathcal{R}_{ABCD}, \Phi) \tag{2.1}$$

where capital Latin indicies run over D-dimensional spacetime ($A, B, ... = 0, 1, ..., D$). G_{AB} is the D-dimensional metric which gives rises to a D-dimensional Riemann tensor $\mathcal{R}_{ABCD}$ and $G = \det G_{AB}$. Φ denotes collectively all other bulk fields.

It should be noted that the volume of the bulk-space of this asymptotically flat model is truly infinite

$$V_{D-4} = \int\ d^{D-4}y\sqrt{-G} \rightarrow \infty \tag{2.2}$$

which differs from that of the volume of the non-compactified bulk of the $5D$ Randall-Sundrum (RSII) model [7] where a non-zero, constant vacuum energy density warps the extra space. Although non-compactified, the physical size of the extra dimension in the RS model is finite due to the warp factor which, consequently, was discovered to allow for a localization of gravity on the 3-brane. The Randall-Sundrum model allows for a modification of the Newtonian potential between two sources on the brane to be of the form

$$V_{RS}(r) = -\frac{GmM}{r}\left(1 + \frac{(2L)^2}{r^2}\right) \tag{2.3}$$

where L is the effective size of the warped extra dimension. From (2.3), it is evident that the second term becomes dominant in the near regime, when $r \lesssim L$; at large r, the first term dominates and we recover $4D$ gravity.

This behavior of the RS model drastically differs from that of the DGP model of a Minkowski 3-brane embedded in a flat, infinite-volume bulk. As will be presented, the DGP model will provide a long range modification of Einstein gravity on the brane where the distance dependence of the potential transitions from that of a $4D$ Newtonian potential in the near regime to that of a higher dimensional theory in the far regime.

In it's present form, (2.1) is invariant under D-dimensional reparametrizations and translations. We wish to embed a 3-brane in the bulk which will break this D-dimensional reparametrizational and translational invariance. Throughout this work, the D-dimensional coordinates will be split as

$$X^A = (x^\mu, y^m) \tag{2.4}$$

where Greek indicies run over the four-dimensional ($4D$) worldvolume coordinates ($\mu, \nu, ... = 0, 1, 2, 3$) and lowercase latin indicies run over the bulk coordinates ($m, n, ... = 4, 5, ..., D$) perpendicular to the brane.

To (2.1) we wish to add additional action contributions which describe an embedded brane residing in the bulk-space. The 3-brane is allowed to contain Standard

Model (SM) matter fields localized on the brane worldvolume. This is consistent with the String theory description which requires open strings, the spin-$(\frac{1}{2},1)$ SM particles, to be confined to the brane dictated by Dirichlet boundary conditions. Conversely, closed strings representing spin-2 gravitons have no such boundary conditions and are free to propagate both in the bulk and on the brane.

The Dirac-Nambu-Goto action and $4D$ localized matter field action takes the form

$$\tilde{S}_{\text{brane}} = -T\int d^4x\sqrt{-g} + \int d^4x\sqrt{-g}\;\mathcal{L}(\phi) \tag{2.5}$$

where the coefficient of the Dirac-Nambu-Goto action, T, is the brane tension. $\mathcal{L}(\phi)$ is the four-dimensional Lagrangian density which is a function of the $4D$ fields ϕ. The tensor $g_{\mu\nu}$ is the induced metric on the brane $g_{\mu\nu} = \partial_\mu X^A \partial_\nu X^B G_{AB}$ whose precise form is dictated by the yet unspecified choice of brane coordinates.

Here we neglect brane fluctuations and the brane is taken to be of the delta-function type, a brane of zero width in the transverse directions. This treatment will later be generalized to allow for a brane of finite thickness. For the delta-function type brane, we choose the location of the brane to be at the origin of our coordinate system, $y_m = 0$. For the case of $5D$, we impose Z_2 symmetry across the brane such that $y \to -y$ around y=0. For this choice of coordinates, the induced metric takes the form

$$g_{\mu\nu}(x^\alpha) = \delta^A_\mu \delta^B_\nu G_{AB}(x^\alpha, 0) \tag{2.6}$$

At this point, the total action which has been presented, $S_{\text{cl}} = S_{\text{bulk}} + \tilde{S}_{\text{brane}}$, is void of a $4D$ Ricci scalar on the brane. Upon solving the equations of motion which arise from the above actions, one would obtain a force law describing that of bulk gravity which scales as $F \sim 1/r^{D-2}$ which would contradict gravitational observations.

The action S_{cl} describes a 3-brane embedded in a bulk at classical level. Additional $4D$ action contributions, if not already present at the classical level, would be generated in a full quantum theory on the brane worldvolume due to the quantum loops of the bulk gravitons interacting with the SM particles residing on the brane. These additional terms should preserve $4D$ reparameterization invariance which will consequently preserve $4D$ gauge invariance along the brane worldvolume. To see the generation of these $4D$ terms explicitly, we note that the localized stress-energy tensor will take the form

$$T_{AB}(x^\alpha, y^m) = \delta^\mu_A \delta^\nu_B \, T_{\mu\nu}(x^\alpha) \, \delta^{D-4}(y^m) \tag{2.7}$$

for the delta-function type brane. The interaction Lagrangian of this localized matter source interacting with the D-dimensional metric fluctuations $h_{AB}(x^\alpha, y^m) = G_{AB}(x^\alpha, y^m) - \eta_{AB}$ takes the following form

$$\mathcal{L}_{\text{int}} = \int d^{D-4}y \; h^{AB}(x^\alpha, y^m) T_{AB} = h^{\mu\nu}(x^\alpha, 0) T_{\mu\nu} \tag{2.8}$$

where we used (2.7). The $4D$ metric fluctuations $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ are defined by (2.6). Due to this interaction of the bulk gravitons with the worldvolume stress-energy tensor, a $4D$ kinetic term can be generated on the brane. The one-loop diagram containing massive scalars and fermions, induce an additional $4D$ action [53, 54, 55] in the low energy action of the form

$$\sim \int d^4x d^{D-4}y \delta^{D-4}(y^m) \sqrt{-g} \mathcal{R}^{(4)} \tag{2.9}$$

In addition to this $4D$ Einstein-Hilbert term, a series in powers of the $4D$ Ricci scalar $\mathcal{R}^{(4)}$ is generated due to higher-order quantum loop corrections. With this being said, the induced action takes the form

$$S_{\text{ind}} = \overline{M}^2 \int d^4x \sqrt{-g} \left[\Lambda + \mathcal{R}^{(4)} + \mathcal{O}(\mathcal{R}^{(4)})^2 + ... \right] \tag{2.10}$$

where the coefficient $\overline{M}^2$ has units of mass^2. Λ is the induced $4D$ cosmological constant and $\mathcal{R}^{(4)}$ is the $4D$ Ricci scalar generated by the $4D$ induced metric tensor $g_{\mu\nu}$. The higher-order Ricci scalar terms are suppressed by powers of $\overline{M}$, which for phenomenological reasons should be on the order of the $4D$ Planck mass scale

$$\overline{M} \sim M_{PL} \simeq 10^{19} GeV \tag{2.11}$$

and will be treated as equal throughout this examination. Combining the induced quantum correction terms to the induced classical action, S_{cl}, we can write the total worldvolume brane action as

$$\begin{aligned} S_{\text{brane}} &= \tilde{S}_{\text{brane}} + S_{\text{ind}} \\ &= -\overline{T} \int d^4x \sqrt{-g} + \overline{M}^2 \int d^4x \sqrt{-g} \left[\mathcal{R}^{(4)} + \mathcal{O}(\mathcal{R}^{(4)})^2 + ...\right] \end{aligned} \tag{2.12}$$

where $\overline{T} = T - \Lambda \overline{M}^2$ is the renormalized brane tension due to the $4D$ cosmological constant. Throughout this present work, we will be interested in solving the field equations for an asymptotically flat $4D$ Minkowski brane, one where $\overline{T} = 0$. For the special case of a $D = 5$ Minkowski bulk, a non-zero positive tension brane inflates [50, 51, 52]. In order to avoid this cosmic inflation and study an asymptotically flat brane, we fine tune the cosmological constant to yield $\overline{T} = 0$.

2.1 The DGP Model

In the previous subsection, we discussed some of the salient features of a general induced braneworld model describing a tensionless, worldvolume brane embedded in an asymptotically flat bulk. In this subsection, we review the pioneering work of

Dvali-Gabadadze and Porrati [8, 9]. Following [8, 9] closely, we limit our examination to that of the simplest setup; that of the bulk Lagrangian consisting of only a D-dimensional Ricci scalar. In the absence of a brane, this D-dimensional Einstein-Hilbert action would give rise to D-dimensional Einstein field equations describing higher-dimensional tensor gravity. We further limit our attention by keeping only the most dominant worldvolume contribution of (2.10) for the case of the fine-tuned brane ($\overline{T} = 0$). This setup is that of the DGP model which includes only the lowest dimensional derivative action contributions in the bulk and on the brane.

To first-order in the Ricci scalars, we have a 3-brane on the boundary of a D-dimensional bulk-space Σ described by the action

$$S_{DGP} = M^{D-2} \int_{\Sigma} d^4x d^{D-4}y \, \sqrt{-G} \, \mathcal{R}^{(D)} + \overline{M}^2 \int_{\partial\Sigma} d^4x \sqrt{-g} \, \mathcal{R}^{(4)} + S_M \qquad (2.13)$$

where G_{AB} is the D-dimensional metric which generates the D-dimensional Ricci scalar $\mathcal{R}^{(D)}$, whereas $\mathcal{R}^{(4)}$ is generated by the four-dimensional metric $g_{\mu\nu}$ which is the induced metric on the brane

$$g_{\mu\nu}(x^\alpha) = \delta^A_\mu \delta^B_\nu G_{AB}(x^\alpha, 0) \qquad (2.14)$$

As was stated in the previous subsection, capital Latin indices run over D-dimensional space-time ($A, B = 0, 1, 2, ..., D$), Greek indices run over the four-dimensional brane worldvolume spanned by coordinates x^μ ($\mu = 0, 1, 2, 3$) and lowercase Latin indices run over the extra space spanned by y_m ($m = 4, 5, ..., D$). S_M is the unspecified matter action.

The coefficient M is the D-dimensional Plank mass. The mass scale $\overline{M}$ is the $4D$ Plank mass and is related to the four-dimensional Newton through the relation

$G_N = 1/8\pi\overline{M}^2$. For the case of the braneworld scenarios residing in a *finite-volume* bulk, the D-dimensional Plank mass is intimately connected to the 4D Plank mass by the relation

$$\overline{M}^2 = M^{D-2}\, V_{D-4} \tag{2.15}$$

For the *infinite-volume* bulk scenario, $\overline{M}$ can in general depend on M, but here they will be treated as independent scales.

We wish to study the effects of the brane induced $4D$ Einstein-Hilbert term on the Minkowski bulk for a delta-function type brane. Applying Hamilton's principle of setting the variation of (2.13) with respect to the tensor field equal to zero, one arrives at the DGP field equations, which are given by

$$M^{D-2}G^{(D)}_{AB}(x^\alpha, y^m) + \overline{M}^2 G^{(4)}_{\mu\nu}(x^\alpha)\ \delta^\mu_A \delta^\nu_B \delta^{D-4}(y^m) = T_{\mu\nu}(x^\alpha)\delta^\mu_A \delta^\nu_B \delta^{D-4}(y^m) \tag{2.16}$$

where $G^{(D)}_{AB}$ ($G^{(4)}_{\mu\nu}$) is the D-dimensional (4D) Einstein tensor. $G^{(4)}_{\mu\nu}$ only has brane worldvolume components of the metric tensor. As was previously stated in (2.7), we have chosen the matter source to be described by a stress-energy tensor $T_{\mu\nu}$ whose transverse components vanish ($T_{mn} = T_{\mu n} = 0$).

Upon examination of (2.16), we note that in the decoupling limit, $\lim_{M\to 0}$, we recover the 4D Einstein field equations on the brane given by

$$\overline{M}^2 G^{(4)}_{\mu\nu}(x^\alpha) = T_{\mu\nu}(x^\alpha) \tag{2.17}$$

Conversely, in the limit of a vanishing brane contribution, $\lim_{\overline{M}\to 0}$, we obtain purely D-dimensional Einstein field equations with a brane matter source.

We are interested in solving the DGP field equations for metric perturbations residing in a Minkowski background. Throughout this treatment we will choose a mostly negative Minkowski metric tensor, $\eta_{AB} = \mathrm{diag}[+--...-]$. Expanding around this flat background,

$$G_{AB} = \eta_{AB} + h_{AB} \tag{2.18}$$

we arrive at the first-order field equations. Due to the 4D reparameterizational invariance, the field equations are instructively split into $\mu\nu$, μn, and nm components. The transverse (nm) components are

$$2\partial^A\partial^n h_{An} - \partial_n\partial^n h^A_A - \partial_A\partial^A h^n_n = (D-4)(\partial^C\partial^D h_{CD} - \partial_C\partial^C h^D_D) \tag{2.19}$$

whereas the mixed (μn) components are

$$\partial_A\partial^A h_{\mu n} = \partial_\mu\partial^A h_{An} + \partial_n\partial^A h_{A\mu} - \partial_\mu\partial_n h^A_A \tag{2.20}$$

Finally, the brane worldvolume ($\mu\nu$) components imply

$$\begin{aligned} M^{D-2}(\partial_\mu\partial^A h_{\nu A} + \partial_\nu\partial^A h_{\mu A} - \partial_A\partial^A h_{\mu\nu} - \partial_\mu\partial_\nu h^A_A &- \eta_{\mu\nu}\left(\partial^A\partial^B h_{AB} - \partial_B\partial^B h^A_A\right)) \\ +\overline{M}^2\delta^{D-4}(y^m)\left(\partial_\mu\partial^\alpha h_{\nu\alpha} + \partial_\nu\partial^\alpha h_{\mu\alpha} - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h^\alpha_\alpha\right. &- \eta_{\mu\nu}\left(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial_\alpha\partial^\alpha h^\beta_\beta\right)) \\ &= T_{\mu\nu}(x^\alpha)\delta^{D-4}(y) \end{aligned} \tag{2.21}$$

Since we are expanding around a Minkowski background, indices are raised and lowered by the flat metric tensor η_{AB}.

Although D-dimensional reparameterizational invariance is broken at the location of the brane, the bulk remains invariant. This implies a D-dimensional gauge invariance in the bulk and the need for D-dimensional gauge fixing. To solve the field equations, we shall choose the harmonic gauge,

$$\partial^A h_{AB} = \frac{1}{2}\partial_B h^A_A \quad . \tag{2.22}$$

Using this harmonic gauge, we obtain from eqs. (2.19) and (2.20), respectively,

$$
\begin{aligned}
(6-D)\partial_A\partial^A h_n^n &= (D-4)\partial_A\partial^A h_\mu^\mu \\
\partial_A\partial^A h_{\mu n} &= 0
\end{aligned} \tag{2.23}
$$

which yields the perturbative relations

$$
h_{\mu n} = 0 \tag{2.24}
$$

$$
h_n^n = \frac{(D-4)}{(6-D)} h_\alpha^\alpha \tag{2.25}
$$

Using (2.24) and (2.25), we can rearrange (2.21) into the form

$$
\begin{aligned}
\left\{T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right\}\delta^{D-4}(y) &= \frac{(D-4)}{(6-D)}\overline{M}^2\delta^{D-4}(y)\partial_\mu\partial_\nu h_\alpha^\alpha \\
&- \left[M^{D-2}\partial_A\partial^A + \overline{M}^2\delta^{D-4}(y)\partial_\mu\partial^\mu\right]\left[h_{\mu\nu} + \frac{(D-5)}{3(6-D)}\eta_{\mu\nu}h_\alpha^\alpha\right]
\end{aligned} \tag{2.26}
$$

where we've written $T = T_\alpha^\alpha$. Taking the trace of (2.26), we arrive at an equation for the scalar propagator given by the the equation

$$
T\delta^{D-4}(y) = \frac{(D-2)}{(6-D)}\left[M^{D-2}\partial_A\partial^A - \frac{2(D-5)}{(D-2)}\overline{M}^2\delta^{D-4}(y)\partial_\mu\partial^\mu\right] h_\alpha^\alpha \tag{2.27}
$$

To solve this equation, we'll Fourier transform the worldvolume coordinates x^α

$$
h_{\mu\nu}(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\tilde{h}_{\mu\nu}(p,y) \tag{2.28}
$$

where for brevity we've dropped the coordinate indicies. Fourier transformed quantitites will be designated by tildes, $h_{\mu\nu} \to \tilde{h}_{\mu\nu}$.

After again rearranging, the equations for the metric perturbations take the form

$$-\left[M^{D-2}(p^2-\Box_N)+\overline{M}^2p^2\delta^{D-4}(y)\right]\left[\tilde{h}_{\mu\nu}+\frac{(D-2)}{3(6-D)}\left(\frac{(D-5)}{(D-2)}\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\tilde{h}^\alpha_\alpha\right] = \left\{\tilde{T}_{\mu\nu}-\frac{1}{3}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\tilde{T}\right\}\delta^{D-4}(y) \tag{2.29}$$

where the Fourier transformed D-dimensional d'Alembertian is $\widetilde{\partial_A\partial^A}=p^2-\partial_n\partial^n=p^2-\Box_N$, $\Box_N$ is the Laplacian of the $(D-4)$-dimensional transverse space. The scalar propagator obeys the Fourier transformed trace equation

$$\left[M^{D-2}(p^2-\Box_N)-\frac{2(D-5)}{(D-2)}\overline{M}^2p^2\delta^{D-4}(y)\right]\tilde{h}^\alpha_\alpha=\frac{(6-D)}{(D-2)}\tilde{T}\delta^{D-4}(y) \tag{2.30}$$

We have written the Fourier transformed equations in terms of the Euclidean momentum

$$p^2=-p_\mu p^\mu=-p_0^2+p_1^2+p_2^2+p_3^2=p_4^2+p_1^2+p_2^2+p_3^2 \tag{2.31}$$

It is instructive to rewrite the expression in eq.(2.29) into the following form

$$\begin{aligned}-\left[M^{D-2}(p^2-\Box_N)+\overline{M}^2p^2\delta^{D-4}(y)\right]\tilde{h}_{\mu\nu}(p,y) &= \left\{\tilde{T}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\tilde{T}\right\}\delta^{D-4}(y)\\ +\frac{(D-4)}{2(6-D)}\eta_{\mu\nu}M^{D-2}(p^2-\Box_N)\tilde{h}^\alpha_\alpha(p,y) &- \frac{(D-4)}{(6-D)}\overline{M}^2p_\mu p_\nu\delta^{D-4}(y)\,\tilde{h}^\alpha_\alpha(p,y)\end{aligned} \tag{2.32}$$

Upon careful examination of the above equation, it should be noted that the first term on the right hand side has the correct tensor structure of 4D Einstein gravity. If the first line of eq.(2.32) comprised the entire equation, one would simply arrive at a solution for a massless or massive 4D propagator depending on the choice of spread function (a delta-function type brane or a brane of finite thickness), for any choice of dimensions. This, however, is not quite the case as there remains two additonal terms on the second line of eq.(2.32).

Eq.(2.32) determines the behavior of the metric perturbations, $\tilde{h}_{\mu\nu}$, which are gauge dependent quantities and vary under the gauge transformation

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{2.33}$$

By making an alternative gauge choice to that of (2.22), one would arrive at a slight variant of (2.32). Note that the convolution of the metric tensor with a conserved stress-energy tensor $\tilde{T}'^{\mu\nu}$ yields a gauge invariant quantity.

$$h_{\mu\nu} T'^{\mu\nu} \longrightarrow h_{\mu\nu} T'^{\mu\nu} + \partial_\mu \xi_\nu T'^{\mu\nu} + \partial_\nu \xi_\mu T'^{\mu\nu} = h_{\mu\nu} T'^{\mu\nu} \tag{2.34}$$

where we used the fact that

$$p_\mu p_\nu \tilde{T}'^{\mu\nu} = 0 \tag{2.35}$$

in Fourier space at tree level.

Convoluting eq.(2.32) with this conserved stress-energy tensor $\tilde{T}'^{\mu\nu}$ yields the gauge invariant equation

$$\begin{aligned} - & \left[M^{D-2}(p^2 - \Box_N) + \overline{M}^2 p^2 \delta^{D-4}(y) \right] \tilde{h}_{\mu\nu} \tilde{T}'^{\mu\nu} \\ = & \left\{ \tilde{T}_{\mu\nu} \tilde{T}'^{\mu\nu} - \frac{1}{2} \tilde{T} \tilde{T}' \right\} \delta^{D-4}(y) + \frac{(D-4)}{2(6-D)} M^{D-2}(p^2 - \Box_N) \tilde{h}^\alpha_\alpha \tilde{T}' \end{aligned} \tag{2.36}$$

On comparison with eq.(2.32), we see that the $p_\mu p_\nu$ term does not remain in the gauge invariant expression and therefore does not contribute at tree level. In addition to the correct combination of the stress-energy tensor which yields 4D Einstein gravity

$$\left\{\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{2}\tilde{T}\tilde{T}'\right\} \tag{2.37}$$

there remains one additional term on the r.h.s. of the above equation due to the trace of the metric perturbations h^{α}_{α}. As expected, this term vanishes when $D = 4$ recovering the expected $4D$ Einstein gravity for the graviton propagator. When $D \neq 4$, the additional trace contribution is non-zero and is of the form

$$M^{D-2}(p^2 - \Box_N)\tilde{h}^{\alpha}_{\alpha}\tilde{T}' \tag{2.38}$$

This term effectively acts as an additional source for the graviton propagator $h_{\mu\nu}T^{\mu\nu}$ and gives rise to an additional scalar exchange yielding $4D$ tensor-scalar gravity.

2.2 Solution for the Graviton and Scalar Propagators

We now proceed with obtaining the solution for the graviton and scalar propagators. Upon examining eqs.(2.29, 2.30), it is evident the solutions can be obtained in terms of a Green function. In addition to yielding the values for the graviton propagator, the solution to the Green function equation corresponds to the exchange of a scalar particle in a 4D world-volume theory and will allow us to calculate the Newtonian potential on the brane.

The Green function needed to solve (2.29, 2.30) is of the form

$$\left[M^{D-2}(p^2 - \Box_N) + \gamma\overline{M}^2 p^2 \delta^{D-4}(y)\right]\tilde{G}(p, y) = \delta^{D-4}(y) \tag{2.39}$$

where γ is a c-number and will take on the value of 1 or $-2(D-5)/(D-2)$. To examine the behavior of the 4D distance dependence of the potential, we'll set $\gamma = 1$ which corresponds to the Green function behavior derivable from a scalar action [8]; the value of $\gamma = -2(D-5)/(D-2)$ becomes important in the tensor analysis. The Fourier transformed Green function $\tilde{G}(p, y)$ is given by the relation

$$G(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \tilde{G}(p, y) \tag{2.40}$$

Once the value of the Green function is obtained, the distance dependence of the interactions is found from a calculation of this potential which is given by [8, 9]

$$V(r) = \int G(t, x^i, y = 0)\, dt \tag{2.41}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the world-volume distance.

To solve eq.(2.39), we choose a product solution of the form

$$\tilde{G}(p, y) = \tilde{B}(p)\tilde{D}(p, y) \tag{2.42}$$

where $\tilde{D}(p, y)$ is defined by

$$(p^2 - \Box_N)\tilde{D}(p, y) = \delta^{D-4}(y) \tag{2.43}$$

This decomposition yields a solution for the Green function of the form

$$\tilde{G}(p, y) = \frac{\tilde{D}(p, y)}{M^{D-2} + \gamma \overline{M}^2 p^2 \tilde{D}(p, 0)} \tag{2.44}$$

For the case of $D > 5$, the function $\tilde{D}(p, y)$, as defined by (2.43), diverges on the brane, at $y = 0$, in this delta-function type brane treatment. In this case, one has a product of two singular functions in (2.39) and (2.42). One should work with the regularized expression $\tilde{G}(p, y) = \lim_{\epsilon \to 0} \tilde{B}(p)\tilde{D}(p, y + \epsilon)$; a careful analysis with this regularization shows that the work presented in this section with these singular functions is valid [9]. An alternative to the aforementioned regularization scheme would be to either keep higher-dimensional derivative terms obtained from the next order contribution to the bulk action (2.1) or to regulate the theory by giving the brane a finite thickness in the transverse direction. In the next chapter we regulate the theory by doing the latter.

The Green function for the delta-function brane in 5 dimensions is well behaved at $y = 0$. Due to this contrasting behavior to the case of $D > 5$, we will proceed to examine the cases $D = 5$ and $D > 5$ separately from this point forward. In the following subsections, we finally present the solutions for the graviton and scalar propagators and arrive at the value of the potential on the brane.

2.2.1 DGP in $D = 5$

For the unique case of $D = 5$, the function $\tilde{D}(p, y)$ in (2.43) is well-behaved on the brane. The solution to (2.39) for the Green function is

$$\tilde{G}(p, y) = \frac{1}{\overline{M}^2} \frac{e^{-py}}{(\gamma p^2 + 2m_b p)} \tag{2.45}$$

where we've imposed Z_2 symmetry across the brane which effectively places the brane at the boundary of the bulk-space and have defined

$$m_b = \frac{M^3}{\overline{M}^2} = \frac{1}{r_b} \tag{2.46}$$

suggestively hinting at a graviton mass. Here we've choosen the positive root of the Euclidean momentum square $p \equiv \pm\sqrt{p^2}$. It should be noted that due to this square root in the Green function, the solution is multi-valued; the choice of sign dictates the particular branch of the theory. At this point we will ignore such subtleties but will return to this in Chapter 4 when we examine the pole structure of the graviton propagator.

Taking the inverse Fourier transform and using eq.(2.41), we arrive at a value for the potential on the brane given by [8]

$$V(r) = \frac{1}{4\pi^2 \overline{M}^2} \cdot \frac{1}{r} \left[\sin\left(\frac{2r}{r_b}\right) \text{ Ci}\left(\frac{2r}{r_b}\right) + \frac{1}{2}\cos\left(\frac{2r}{r_b}\right) \left\{ \pi - 2\text{Si}\left(\frac{2r}{r_b}\right) \right\} \right] \qquad (2.47)$$

where $\text{Si}(z),\ \text{Ci}(z)$ are the sine and cosine integrals given by the relations

$$\begin{aligned} \text{Ci}(z) &= \gamma + \ln(z) + \int_0^z \frac{\cos(t) - 1}{t} dt \\ \text{Si}(z) &= \int_0^z \frac{\sin(t)}{t} dt \\ \bar{\gamma} &\simeq .577 \end{aligned} \qquad (2.48)$$

for $\gamma = 1$ where $\bar{\gamma}$ is the Euler-Masceroni constant. As is obvious from (2.47), the potential is dependent on the ratio r/r_b where r_b is the distance scale defined in (2.46). As will soon become obvious, r_b is a critical radius where the distance dependence of the potential crosses over from that of 4D behavior to that of a 5D theory. This can be directly witnessed from an expansion of the sine and cosine integrals.

In the near regime ($r \ll r_b$), the potential behaves as

$$V(r) \simeq \frac{1}{4\pi^2 \overline{M}^2} \cdot \frac{1}{r} \left[\frac{\pi}{2} + \left\{ -1 + \bar{\gamma} + \ln\left(\frac{r}{r_b}\right) \right\} \left(\frac{r}{r_b}\right) + \mathcal{O}(\frac{r^2}{r_b^2}) \right] \qquad (2.49)$$

The potential scales as $1/r$ with an additional logarithmic repulsion term. This logarithmic repulsion predicts a deviation from 4D Newtonian gravity which is dependent on the value of the r_b.

In the far regime ($r \gg r_b$), we obtain

$$V(r) \simeq \frac{1}{4\pi^2 \overline{M}^2} \cdot \frac{1}{r} \left[\frac{r_b}{r} + \mathcal{O}(\frac{r_b^2}{r^2}) \right] \tag{2.50}$$

which to first order corresponds to a potential of a $5D$ theory.

Using the solution for the Green function given in (2.45), we can obtain the solution for the metric and scalar perturbations (2.29, 2.30) in $D = 5$. The solutions are

$$\begin{aligned} \tilde{h}_{\mu\nu}(p,y) &= -\frac{1}{\overline{M}^2(p^2 + 2m_b p)} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{3} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{2m_b p} \right) \tilde{T} \right\} e^{-py} \\ \tilde{h}^\alpha_\alpha(p,y) &= \frac{\tilde{T}}{6\overline{M}^2 m_b p} e^{-py} \end{aligned} \tag{2.51}$$

We can examine this solution on the brane in the decoupling limit ($y = 0,\ \lim_{m_b \to 0}$) and compare it to that of the 4D Einstein solution given by the expressions

$$\begin{aligned} \tilde{h}^{4D}_{\mu\nu}(p) &= -\frac{1}{\overline{M}^2 p^2} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{T} \right\} \\ \tilde{h}^{\alpha\ 4D}_\alpha(p) &= \frac{\tilde{T}}{\overline{M}^2 p^2} \end{aligned} \tag{2.52}$$

In this decoupling limit, one would expect (2.51) to reduce to (2.52) on the brane. As is obvious from (2.51), this is certainly not the case. In the decoupling limit, note that the factor of 1/3 remains. This discrepancy with the factor of 1/2 of the 4D Einstein solution signals a discontinuity and is known to disagree with astronomical data. Also notice that both the scalar propagator and the $p_\mu p_\nu$ term diverge in this limit, one should certainly demand a small value of $\tilde{h}_{\mu\nu},\ \tilde{h}^\alpha_\alpha$ in a valid perturbative solution. We will come back to these discrepancies in Chapters 4 and 5 when we show that they are do to a breakdown of the linearized theory and obtain solutions which are well-behaved in the decoupling limit.

2.2.2 DGP in $D > 5$

As was previously mentioned, when $D > 5$ the function $\tilde{D}(p, y)$ in (2.43) diverges on the brane in the expression for the Green function

$$\tilde{G}(p, y) = \frac{\tilde{D}(p, y)}{M^{D-2} + \gamma \overline{M}^2 p^2 \tilde{D}(p, 0)} \tag{2.53}$$

This expression for the Green function is discontinuous at the location of the brane, yielding a finite jump between the solution on the brane world-volume ($y = 0$) and in the bulk ($y \neq 0$). Due to this divergence, we obtain the discontinuous Green function of the form

$$\tilde{G}(p, 0) = \frac{1}{\gamma \overline{M}^2 p^2} \tag{2.54}$$

on the brane. Taking the inverse Fourier transform and using eq.(2.41), we arrive at a value for the potential on the brane worldvolume of

$$V(r) = \frac{1}{8\pi \overline{M}^2} \cdot \frac{1}{r} \tag{2.55}$$

which is exactly the static potential between two point sources of unit mass in a purely four-dimensional theory.

Using eqs.(2.36), we obtain a value for the metric perturbations of the form

$$\begin{aligned} \tilde{h}_{\mu\nu}(p, 0) \tilde{T}'^{\mu\nu} &= -\frac{1}{\overline{M}^2 p^2} \left\{ \tilde{T}_{\mu\nu} \tilde{T}'^{\mu\nu} - \frac{1}{2} \tilde{T} \tilde{T}' \right\} \\ \tilde{h}^{\alpha}_{\alpha}(p, 0) &= -\frac{(6 - D)}{2(D - 5)} \cdot \frac{\tilde{T}}{\overline{M}^2 p^2} \end{aligned} \tag{2.56}$$

Comparing this solution for the graviton propagator on the brane to that of 4D Einstein gravity given by (2.52), we find identical expressions. Therefore, the DGP model for a delta-function type brane in $D > 5$ yields a graviton propagator with an identical tensor structure and 4D momentum dependence to that of a purely 4D theory. As was mentioned earlier in this section, this expression is obtained in a singular manner and should be treated carefully.

Although we obtain identical expressions for the graviton propagator, it should be noted that we acquire a value for the trace, $\tilde{h}^{\alpha}_{\alpha}$, which agrees in the 4D momentum dependence but differs by a multiplicative factor. This discrepancy can be accounted for by noting that the scalar propagator $\tilde{h}^{\alpha}_{\alpha}$ is not a gauge-invariant quantity unlike the graviton propagator $\tilde{h}_{\mu\nu}(p,0)\tilde{T}'^{\mu\nu}$. By making a different choice of gauge, one should be able to scale the value of the scalar propagator to within agreement of the 4D theory.

In the bulk ($y \neq 0$), there are two distinct cases which differ depending on the value of the four-momentum squared. The Green function for an identically zero four-momentum squared ($p^2 = 0$) differs from that of the a non-vanishing four-momentum squared. The two distinct cases are presented in turn.

Vanishing four-momentum squared ($p^2 = 0$)

In this case, (2.39) and (2.43) reduce to the form

$$\begin{aligned} -M^{D-2}\Box_N \tilde{G}(0,y) &= \delta^{D-4}(y) \\ -\Box_N \tilde{D}(0,y) &= \delta^{D-4}(y) \end{aligned} \tag{2.57}$$

which is the equation for the Euclidean Green function in the transverse space. The solution for the Green function is given by the expression

$$\tilde{G}(0,y)|_{y\neq 0} \sim \lim_{p^2\to 0} \frac{1}{M^{D-2}} \left(\frac{p}{y}\right)^{(D-6)/2} K_{(D-6)/2}(py) \tag{2.58}$$

Where $K_n(py)$ is a modified Bessel function. The solution scales as

$$\tilde{G}(0,y)|_{D>6} \sim \frac{1}{y^{D-6}} \tag{2.59}$$

When $D = 6$, the Euclidean Green function has a logarithmic singularity at $p^2 = 0$

$$\tilde{G}(0,y)|_{D=6} \sim \ln(py)|_{p^2=0} \to \infty \tag{2.60}$$

Therefore, there exists a $(D-4)$-dimensional Green function for the case when $p^2 = 0$ which corresponds to the $p^2 = 0$ mode providing interactions between the bulk and brane; this indicates that the bulk-space exhibits infrared transparency and can be probed by gravitons of vanishing four-momentum. This mode should be produced with a non-zero three-momentum.

Again, using (2.29,2.30), the solution for the metric perturbations are

$$\begin{aligned} \tilde{h}_{\mu\nu}(0,y) &= -\frac{1}{M^{D-2}} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{(D-2)} \eta_{\mu\nu} \tilde{T} \right\} \tilde{D}(0,y) \\ \tilde{h}^{\alpha}_{\alpha}(0,y) &= \frac{1}{M^{D-2}} \frac{(6-D)}{(D-2)} \tilde{D}(0,y) \end{aligned} \tag{2.61}$$

This solution for the metric perturbations have a D-dimensional tensor structure and a $(D-4)$-dimensional distance dependence; this is the solution of D-dimensional theory. For the case of $(p^2 = 0)$, the metric perturbations give rise to interactions between matter placed in the bulk and matter localized on the brane [9].

Non-vanishing four-momentum squared ($p^2 \neq 0$)

For the case of $p^2 \neq 0$, the Green function, and hence the metric perturbations, vanish identically in the bulk

$$\tilde{G}(p, y)|_{y\neq 0} = 0$$
$$\tilde{h}_{\mu\nu}(p, y)|_{y\neq 0} = 0 \tag{2.62}$$

Therefore, the $p^2 \neq 0$ mode cannot produce interactions between the bulk and brane worldvolume matter.

To restate the results of the chapter, we reexamined the general features of Brane-Induced-Gravity of a 3-brane embedded in an infinite-volume, Minkowski bulk-space. We limited our analysis to the simplest case of the Dvali-Gabadadze-Porrati (DGP) model which is described by an induced 4D Einstein-Hilbert action, which arises from the interactions of the brane worldvolume matter with the bulk gravitons, of a *delta-function type* 3-brane embedded in a Minkowski bulk-space described by a D-dimensional Einstein-Hilbert action. Expanding around the flat background, we obtain the solution for the graviton propagator in the separate cases of $D = 5$ and $D > 5$. For the case of $D = 5$, we show that the solution suffers from the vDVZ discontinuity of massive gravity and does not reproduce a 4D Newtonian potential in the decoupling limit. This vDVZ discontinuity is due to the breakdown of the weak field expansion; we will return to this problem in Chapters 4 and 5.

For the case of $D > 5$, we obtain the solution for the graviton propagator which is discontinuous at the location of the brane. This discontinuity of the solution for the graviton propagator can be made continuous by giving the brane an effective thickness which regulates the theory. On the brane, we show that the graviton propagator is exactly that of 4D Einstein gravity yielding a 4D tensor structure and 4D distance dependence. In the bulk, we obtain a non-zero amplitude for a vanishing

4D worldvolume momentum only, which implies that the bulk-space exhibits infrared transparency. In the next chapter, we proceed by studying a “fat” brane in $D > 5$ dimensions, which is much more involved than the delta-function type brane, to regulate the discontinuity of the graviton propagator. After the solution for the graviton propagator in the fat brane scenerio is found, we should be able to proceed by taking a thin brane limit of the graviton propagator and reproduce the results of the DGP model of the delta-function type brane.

Chapter 3

Fat Branes in Infinite-Volume Extra Space

In the previous chapter, we discussed the DGP model of a *delta-function* type 3-brane residing in a Minkowski bulk. In this chapter, the 3-brane is allowed to have finite thickness into the bulk-space with extent governed by the density function $\sigma_\Lambda(y)$. This generalized "fat" brane scenario can arise from transverse fluctuations of the δ-function brane into the bulk giving the brane an effective finite thickness. Alternatively, and from a more fundamental perspective, a brane of non-zero thickness can arise as a smooth soliton solution to a higher-dimensional theory. To see this explicitly, it is instructive to consider ϕ^4 interactions for a scalar particle in a five-dimensional field theory. [1] This scenario can be described by a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2}\partial_A\phi\partial^A\phi - \frac{1}{2}\lambda(\phi^2 - \eta^3)^2 \tag{3.1}$$

The above Lagrangian is invariant under the $\mathbf{Z}_2$ transformation $\phi \to -\phi$, however, the vacuum states of the field are not. The symmetry breaking of the $\mathbf{Z}_2$ transformation

[1] See [27] for a more detailed analysis from which the above discussion is extracted

is due to the fact that the vacuum states $\phi_0 = \pm\eta^{3/2}$ interchange under $\mathbf{Z}_2$. [2] This symmetry breaking suggests that a domain wall should exist. The classical equation of motion, which arises from the above Lagrangian, yields a domain wall solution of the form

$$\phi_{cl}(y) = \eta^{3/2} \tanh\left(\sqrt{\lambda}\, \eta^{3/2} y\right) \tag{3.2}$$

which has a kink at the origin ($y = 0$).

As was shown in the previous chapter, the Green function for a delta-function type brane is discontinuous at the boundary where the brane resides for $D > 5$. This is due to the fact that the solution for the Green function is given by the product solution $\tilde{G}(p, y) = \tilde{B}(p)\tilde{D}(p, y)$ where the function $D(p, y)$, defined by the equation

$$(p^2 - \Box_N)\tilde{D}(p, y) = \delta^{D-4}(y) \tag{3.3}$$

diverges at $y = 0$ when $D > 5$. A regularization scheme of either keeping higher-dimensional derivative terms obtained from the next order contribution to the bulk action (2.1), introducing a UV cutoff into the theory, or to alternatively regulate the theory by giving the brane a finite thickness into the transverse direction [22, 23, 25, 26, 27, 28, 29] can be adopted which will allow for a careful examination of the behavior of this Green function in the delta-function brane limit. For a brane of finite thickness, the solution for the Green function will remain continuous at the boundary.

[2] It should be noted here that the vacuum state ϕ_0 is a multi-valued function with the complex plane containing two Riemann sheets; we'll return to multi-valued functions in Chapter 4 when we discuss the poles of the graviton propagator when we consider the Constrained Perturbative Expansion.

3.1 D-Dimensional Fat Brane Model

We are interested in the dynamics of a 3-brane in a D-dimensional infinite space. The action for Brane Induced Gravity of a 3-brane of finite thickness is similar to the one discussed in the previous chapter (2.13), [9]

$$S = M^{D-2} \int d^4x d^{D-4}y \sqrt{-G}\, \mathcal{R}^{(D)} + \overline{M}^2 \int d^4x d^{D-4}y \sqrt{-g}\, \sigma_\Lambda(y) \mathcal{R}^{(4)} + S_M \quad (3.4)$$

where, as in the original DGP treatment, G_{AB} is the D-dimensional metric which generates the D-dimensional Ricci scalar $\mathcal{R}^{(D)}$, whereas $\mathcal{R}^{(4)}$ is generated by the four-dimensional metric $g_{\mu\nu}$ which is the induced metric on the slice $\vec{y} = \text{const}$.

$$g_{\mu\nu}(x^\alpha, y^m)|_{|\vec{y}|=const} = \delta^A_\mu \delta^B_\nu G_{AB}(x^\alpha, y^m)|_{|\vec{y}|=const} \quad (3.5)$$

Capital Latin indices run over D-dimensional space-time ($A, B = 0, 1, 2, ..., D-1$), Greek indices run over the four-dimensional brane worldvolume spanned by coordinates x^μ ($\mu = 0, 1, 2, 3$) and lowercase Latin indices run over the extra space spanned by y_m ($m = 4, 5, ..., D$). M is the D-dimensional Plank mass. S_M is the unspecified matter action giving rise to the fat brane configuration.

We set $y = |\vec{y}| = \sqrt{y_1^2 + y_2^2 + ... + y_{D-4}^2}$. The density function $\sigma_\Lambda(y)$ is a smooth function of width $1/\Lambda$ which approximates a δ-function and generically obeys the following relations

$$\int d^{D-4}y \sigma_\Lambda(y) = 1 \ , \ \lim_{\Lambda\to\infty} \sigma_\Lambda = \delta^{D-4}(y) \quad (3.6)$$

For explicit calculations, we will choose a step-function form of the density σ_Λ,

$$\sigma_\Lambda(y) = (D-4)\frac{\Lambda^{D-4}}{\omega_{D-4}}\, \Theta(1/\Lambda - y) \quad (3.7)$$

where ω_n is the surface area of the unit n-dimensional sphere. The careful reader may wish to smooth the step-function first and then take the limit in which σ_Λ becomes discontinuous. Our results are not altered.

Let us take a closer look at the structure of the brane contribution to the action. This continuous distribution of the 3-brane may be thought of as the continuous limit of a discrete set of the four-dimensional, delta-function type hypersurfaces (infinitely thin 3-branes), as discussed in [9], each placed at position y_i

$$S_{\text{brane}} = \overline{M}^2 \sum_{i=1}^{N'} \int d^4x \sqrt{-g_i(x,y)}\, \mathcal{R}_i^{(4)}(x,y)|_{|\vec{y}|=const} \tag{3.8}$$

where the 4D Ricci scalar and metric tensor are evaluated at position y_i. Allowing the delta-function type branes to be 'smoothed out' in the y direction, we obtain

$$\sum_{i=1}^{N'} \sqrt{-g_i(x,y)}\mathcal{R}_i^{(4)}(x,y)|_{|\vec{y}|=const} \rightarrow \int d^{D-4}y\, \sqrt{-g}\sigma_\Lambda(y)\mathcal{R}^{(4)}(x,y) \tag{3.9}$$

This treatment is in close analogy to electromagnetic theory where point charges are replaced by a charge density and the sum replaced by an integral in the limit of the charges becoming very small and numerous.

Upon varying the action (3.4), we arrive at the field equations which are given by

$$M^{D-2}G_{AB}^{(D)}(x^\alpha, y^m) + \overline{M}^2 G_{\mu\nu}^{(4)}(x^\alpha, y^m)\delta_A^\mu \delta_B^\nu \sigma_\Lambda(y) = T_{AB}(x^\alpha, y^m) \tag{3.10}$$

where $G_{AB}^{(D)}$ is the D-dimensional Einstein tensor and $G_{\mu\nu}^{(4)}$ only has brane worldvolume components. Expanding around a flat background,

$$G_{AB} = \eta_{AB} + h_{AB} \tag{3.11}$$

the first-order Einstein equations are as follows. The trace of the transverse components (nm) give

$$2\partial^A\partial^n h_{An} - \partial_n\partial^n h^A_A - \partial_A\partial^A h^n_n = (D-4)(\partial^C\partial^D h_{CD} - \partial_C\partial^C h^D_D) \tag{3.12}$$

The mixed components (αn) give

$$\partial_A\partial^A h_{\alpha n} = \partial_\alpha\partial^A h_{An} + \partial_n\partial^A h_{A\alpha} - \partial_\alpha\partial_n h^A_A \tag{3.13}$$

and the brane worldvolume components $(\alpha\beta)$ imply

$$\begin{aligned} M^{D-2}(\partial_\alpha\partial^A h_{\beta A} + \partial_\beta\partial^A h_{\alpha A} - \partial_A\partial^A h_{\alpha\beta} - \partial_\alpha\partial_\beta h^A_A &- \eta_{\alpha\beta}\left(\partial^A\partial^B h_{AB} - \partial_B\partial^B h^A_A\right)) \\ +\overline{M}^2\sigma_\Lambda(y)\,(\partial_\alpha\partial^\nu h_{\beta\nu} + \partial_\beta\partial^\nu h_{\alpha\nu} - \partial_\nu\partial^\nu h_{\beta\alpha} - \partial_\alpha\partial_\beta h^\nu_\nu &- \eta_{\alpha\beta}\,(\partial^\mu\partial^\nu h_{\mu\nu} - \partial_\mu\partial^\mu h^\nu_\nu)\,) \\ &= T_{\alpha\beta}(x^\mu, y^m) \end{aligned} \tag{3.14}$$

where we have chosen a matter source described by the stress-energy tensor $T_{\mu\nu}$ whose transverse components vanish ($T_{mn} = T_{\mu n} = 0$). Indices are raised and lowered by the flat metric tensor η_{AB}.

To solve the field equations, we shall choose the harmonic gauge,

$$\partial^A h_{AB} = \frac{1}{2}\partial_B h^A_A \quad . \tag{3.15}$$

We obtain from eqs. (3.12) and (3.13), respectively,

$$(6-D)\partial_A\partial^A h^n_n = (D-4)\partial_A\partial^A h^\mu_\mu \tag{3.16}$$

$$\partial_A\partial^A h_{m\alpha} = 0 \tag{3.17}$$

so we may set

$$h_{m\alpha} = 0 \tag{3.18}$$

$$(D-6)h_n^n + (D-4)h_\mu^\mu = 0 \tag{3.19}$$

Then the brane worldvolume components of the field equations can be written in the following form:

$$-M^{D-2}\partial_A\partial^A\left(h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}h_B^B\right) + \overline{M}^2\sigma_\Lambda\left(-\partial_\nu\partial^\nu h_{\alpha\beta} + \partial_\alpha\partial_\beta h_n^n - \tfrac{1}{2}\eta_{\alpha\beta}\,\partial^\mu\partial_\mu(h_n^n - h_\nu^\nu)\right) = T_{\alpha\beta}(x^\mu, y^m) \tag{3.20}$$

Performing a Fourier transform in the brane worldvolume coordinates x^μ and multiplying by an arbitrary conserved stress-energy tensor $T'^{\alpha\beta}$, which for simplicity is assumed to have no $\vec{y}$-dependence, we obtain

$$\tilde{T}_{\alpha\beta}(p^\mu, y^m)\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}'^\mu_\mu\left[\overline{M}^2p^2\sigma_\Lambda\,(\tilde{h}_\nu^\nu - \tilde{h}_n^n) + M^{D-2}(p^2 - \Box_N)\tilde{h}_A^A\right] = -\left[M^{D-2}(p^2 - \Box_N) + \overline{M}^2p^2\sigma_\Lambda\right]\tilde{h}_{\alpha\beta}\tilde{T}'^{\alpha\beta} \tag{3.21}$$

where the Fourier transformed, D-dimensional d'Alembertian is $\widetilde{\partial_A\partial^A} = p^2 - \Box_N$ with $p^2 = p_4^2 + \vec{p}^2$ the worldvolume Euclidean four-momentum and $\Box_N$ the $(D-4)$-dimensional Laplacian operator. In the next section, we procede to solve this equation for the graviton propagator.

3.2 Graviton Propagator

In general, the spread functions of the brane and the matter source are different. However, it was argued by Dvali, *et al.* [24] that the two spreads coincide at lowest order with correction terms suppressed by factors $\mathcal{O}(M/\overline{M})$. We shall therefore adopt a source stress-energy tensor of the form

$$T_{\alpha\beta}(x^\mu, \vec{y}) = T_{\alpha\beta}(x^\mu)\, \sigma_\Lambda(y) \tag{3.22}$$

in the explicit calculation of the tensor structure and momentum dependence of the graviton propagator.

Taking the trace of eq. (3.20), we obtain

$$\frac{(D-2)}{(6-D)} \left[M^{D-2}(p^2 - \Box_N) - \frac{2(D-5)}{(D-2)} \overline{M}^2 p^2 \sigma_\Lambda \right] \tilde{h}^\alpha_\alpha = \tilde{T}^\alpha_\alpha\, \sigma_\Lambda(y) \tag{3.23}$$

where we used eq. (3.19) to express $\tilde{h}^n_n$ in terms of $\tilde{h}^\alpha_\alpha$. This is an equation for the field $\tilde{h}^\alpha_\alpha$ (trace over transverse directions of the metric field), which is a scalar from a four-dimensional point of view. The solution is obtained on the brane and in the bulk in terms of the Green function to the wave equation,

$$\left[M^{D-2}\,(p^2 - \Box_N) - (\lambda - 1)\, \overline{M}^2 p^2 \sigma_\Lambda \right] \mathcal{G}_\lambda(p, y) = \sigma_\Lambda(y) \tag{3.24}$$

as

$$\tilde{h}^\alpha_\alpha(p, y) = \frac{(6-D)}{(D-2)}\, \tilde{T}^\alpha_\alpha\, \mathcal{G}_\lambda(p, y) \quad , \quad \lambda = \frac{3(D-4)}{(D-2)} \tag{3.25}$$

After some algebra (see Appendix), we obtain a spherically symmetric solution expressed in terms of Bessel functions as

$$\mathcal{G}_\lambda(p, y)\,|_{y \le 1/\Lambda} = -\frac{1}{(\lambda - 1)\overline{M}^2 p^2} \left[1 + \frac{1}{\mathcal{A}_\lambda} \left(\frac{1}{y\Lambda} \right)^{(D-6)/2} K_{(D-4)/2}(p/\Lambda) J_{(D-6)/2}(k_\lambda p y) \right] \tag{3.26}$$

inside the brane ($y \leq 1/\Lambda$), and

$$\mathcal{G}_\lambda(p,y)\mid_{y>1/\Lambda} = -\frac{1}{(\lambda-1)\overline{M}^2 p^2}\cdot\frac{k_\lambda}{\mathcal{A}_\lambda}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} J_{(D-4)/2}(k_\lambda p/\Lambda)K_{(D-6)/2}(py) \tag{3.27}$$

in the bulk ($y > \Lambda$), where

$$\mathcal{A}_\lambda = k_\lambda K_{(D-6)/2}(p/\Lambda)J_{(D-4)/2}(k_\lambda p/\Lambda) - K_{(D-4)/2}(p/\Lambda)J_{(D-6)/2}(k_\lambda p/\Lambda) \tag{3.28}$$

and we have introduced the constant k_λ given by

$$k_\lambda^2 = (\lambda-1)\,\frac{(D-4)\Lambda^{D-4}\,\overline{M}^2}{\omega_{D-4}\,M^{D-2}} - 1 \simeq (\lambda-1)\,\frac{(D-4)\Lambda^{D-4}\,\overline{M}^2}{\omega_{D-4}\,M^{D-2}} \tag{3.29}$$

Notice that inside the brane, $\mathcal{G}_\lambda(p,y)$ oscillates rapidly over the transverse width of the brane.

To obtain the graviton propagator, we will also need the Green function which is the solution to eq. (3.24) when $\lambda = 0$. Notice that when $\lambda = 0$, eq. (3.24) turns into the wave equation for a scalar field in the thin-brane limit. This scalar field equation is derivable from a scalar field action. The solution for the Green function when $\lambda = 0$ is given in terms of Modified Bessel functions (see Appendix) of the form

$$\mathcal{G}_0(p,y)\mid_{y\leq 1/\Lambda} = \frac{1}{\overline{M}^2 p^2}\left[1 - \frac{1}{\mathcal{A}_0}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} K_{(D-4)/2}(p/\Lambda)I_{(D-6)/2}(\kappa p y)\right] \tag{3.30}$$

inside the brane ($y \leq 1/\Lambda$), and

$$\mathcal{G}_0(p,y)\mid_{y>1/\Lambda} = \frac{1}{\overline{M}^2 p^2}\cdot\frac{\kappa}{\mathcal{A}_0}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} I_{(D-4)/2}(\kappa p/\Lambda)K_{(D-6)/2}(py) \tag{3.31}$$

in the bulk ($y > 1/\Lambda$), where

$$\mathcal{A}_0 = \kappa I_{(D-4)/2}(\kappa p/\Lambda)K_{(D-6)/2}(p/\Lambda) + I_{(D-6)/2}(\kappa p/\Lambda)K_{(D-4)/2}(p/\Lambda) \tag{3.32}$$

and (see eq. (3.29))

$$\kappa^2 = -k_0^2 = 1 + (D-4)\frac{\Lambda^{D-4}\,\overline{M}^2}{\omega_{D-4}\,M^{D-2}} \simeq (D-4)\frac{\Lambda^{D-4}\,\overline{M}^2}{\omega_{D-4}\,M^{D-2}} \tag{3.33}$$

We are now ready to deduce the full graviton propagator. To this end, let us massage eq. (3.21) into the form

$$\left[M^{D-2}(p^2 - \Box_N) + \overline{M}^2 p^2 \sigma_\Lambda(y)\,\right] \left\{\tilde{h}_{\alpha\beta}(p,y)\tilde{T}'^{\alpha\beta} + \frac{(D-5)}{3(6-D)}\tilde{h}^{\alpha}_{\alpha}(p,y)\tilde{T}'^{\nu}_{\nu}\right\}$$

$$= -\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{3}\tilde{T}^{\mu}_{\mu}\tilde{T}'^{\nu}_{\nu}\right\}\sigma_\Lambda(y) \tag{3.34}$$

The solution for the graviton propagator is readily obtained in terms of the scalar propagators,

$$\tilde{h}_{\alpha\beta}(p,y)\tilde{T}'^{\alpha\beta} = -\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{3}\tilde{T}^{\mu}_{\mu}\tilde{T}'^{\nu}_{\nu}\right\}\mathcal{G}_0(p,y) - \frac{(D-5)}{3(D-2)}\tilde{T}^{\mu}_{\mu}\tilde{T}'^{\nu}_{\nu}\mathcal{G}_\lambda(p,y) \tag{3.35}$$

where we used eqs. (3.24) and (3.25). This is the solution for the graviton propagator for a brane of finite thickness which should reduce to that of the solution for the graviton propagator for the delta-function type brane when $\Lambda \to \infty$. Throughout the remainder of the chapter we shall analyze the 4D momentum dependence, the pole structure, and the Λ dependence of the propagator.

3.3 Poles of the Graviton Propagator

Next, we analyze the pole structure of the graviton propagator. We then compare the results of our model with that of Dubovsky and Rubakov [29].

3.3.1 Our model

Using the expressions (3.30) for $\mathcal{G}_0(p,y)$ and (3.26) for $\mathcal{G}_\lambda(p,y)$, the graviton propagator (3.35) inside the brane ($y \leq 1/\Lambda$) can be written in the form

$$\tilde{h}_{\alpha\beta}(p,y)\tilde{T}'^{\alpha\beta} = -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta\right\} \quad + \quad \left(\frac{1}{y\Lambda}\right)^{(D-6)/2} K_{(D-4)/2}(p/\Lambda)$$
$$\times \quad \frac{1}{\overline{M}^2 p^2}\left[\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{3}\tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta\right\}\frac{1}{\mathcal{A}_0} I_{(D-6)/2}(\kappa p y) + \frac{1}{6}\tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta \frac{1}{\mathcal{A}_\lambda} J_{(D-6)/2}(k_\lambda p y)\right] \tag{3.36}$$

For convenience, we have separated the term that corresponds to the tensor structure and momentum dependence of the four-dimensional graviton propagator. To study the pole structure, we shall introduce the average value of the graviton propagator over the transverse directions of the brane (see [23] for problems associated with the definition of observables on the brane) defined by

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p) = \int d^{D-4}y\,\sigma_\Lambda(y)\tilde{h}_{\alpha\beta}(p,y) \tag{3.37}$$

Integrating (3.36), we obtain

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p)\tilde{T}'^{\alpha\beta} = -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta\right\}$$
$$- \quad (D-4)(6-D)\left(\frac{\Lambda}{\overline{M}p^2}\right)^2\left[\frac{T^2_{1/3}}{\kappa^2[1+\mu_0(\kappa p/\Lambda)]} + \frac{\frac{1}{6}\tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta}{k^2[1-\mu_\lambda(k_\lambda p/\Lambda)]}\right] \tag{3.38}$$

where for brevity we wrote $T_{1/3}^2 = \left\{ \tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{3}\tilde{T}_\alpha^\alpha \tilde{T}'^\beta_\beta \right\}$. We may easily deduce the pole structure of the graviton propagator inside the brane. The above expression is valid for $D > 6$ (for $D = 6$, we obtain logarithmic corrections, but the results are similar and will not be explicitly discussed here). The functions that appear in the denominators in (3.38) are

$$\mu_0(z) = \frac{D-6}{z} \frac{I_{(D-6)/2}(z)}{I_{(D-4)/2}(z)} \quad , \quad \mu_\lambda(z) = \frac{D-6}{z} \frac{J_{(D-6)/2}(z)}{J_{(D-4)/2}(z)} \tag{3.39}$$

for $D > 6$. The poles of the propagator are solutions to the equations

$$\mu_0(\kappa p/\Lambda) = -1 \quad , \quad \mu_\lambda(k_\lambda p/\Lambda) = 1 \tag{3.40}$$

Using (3.39) and the Bessel function identity

$$zJ_{\nu-1}(z) + zJ_{\nu+1}(z) = 2\nu J_\nu(z) \tag{3.41}$$

for $\nu = (D-6)/2$, it is easily shown that the solutions to $\mu_\lambda(z) = 1$ are the roots of $J_{\nu-1} = J_{(D-8)/2}$. As is well-known, there are infinitely many zeros for $\nu > 0$, i.e., $D > 6$, which is the case we are considering here. We shall denote them by z_j,

$$J_{(D-8)/2}(z_j) = 0 \quad , \quad j = 1, 2, \dots \tag{3.42}$$

We therefore obtain an infinite tower of tachyonic poles with masses given by

$$m_{*j}^2 = p_{*j}^2 = z_j^2 \frac{\Lambda^2}{k_\lambda^2} \tag{3.43}$$

Similarly, the condition $\mu_0(z) = -1$, together with the Bessel function identity

$$zI_{\nu-1}(z) - zI_{\nu+1}(z) = 2\nu I_\nu(z) \tag{3.44}$$

and the relation $I_\nu(z) = e^{-\pi\nu i/2} J_\nu(iz)$, lead to a tower of massive poles with masses given by

$$m_j^2 = -p_j^2 = -z_j^2 \frac{\Lambda^2}{\kappa^2} \tag{3.45}$$

To obtain the behavior of the propagator near a massive pole, observe that

$$1 + \mu_0(z) = \frac{I_{(D-8)/2}(z)}{I_{(D-4)/2}(z)} = \frac{I'_{(D-8)/2}(z_j)}{I_{(D-4)/2}(z_j)} (z - z_j) + \mathcal{O}((z - z_j)^2) \tag{3.46}$$

Using the Bessel function identity

$$zI'_{\nu-1}(z) = (\nu - 1) I_{\nu-1}(z) + zI_\nu(z) \tag{3.47}$$

together with (3.44), we deduce

$$1 + \mu_0(z) = \frac{1}{(D-6)} (z^2 - z_j^2) + \ldots \tag{3.48}$$

near $z = z_j$. It follows that the graviton propagator on the brane (3.38) behaves as

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p) \tilde{T}'^{\alpha\beta} \sim -(D-4) \frac{(D-6)^2 / z_j^4}{\overline{M}^2 (p^2 + m_j^2)} \left\{ \tilde{T}_{\alpha\beta} \tilde{T}'^{\alpha\beta} - \frac{1}{3} \tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta \right\} \tag{3.49}$$

near the massive pole $p^2 = -m_j^2$. Similarly, near the tachyonic pole $p^2 = -m_{*j}^2$, we obtain

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p) \tilde{T}'^{\alpha\beta} \sim +\frac{(D-4)}{6} \frac{(D-6)^2 / z_j^4}{\overline{M}^2 (p^2 - m_{*j}^2)} \tilde{T}^\alpha_\alpha \tilde{T}'^\beta_\beta \tag{3.50}$$

The plus sign of the residue of the tachyon implies that the tachyon is a ghost.

Notice that both the massive modes (3.45) and the tachyons (3.43) are expressed in terms of the same mass scale parameter p_c, where

$$p_c^2 \sim \frac{\Lambda^2}{\kappa^2} \sim \frac{\Lambda^2}{k_\lambda^2} \sim \frac{M^{D-2}}{\overline{M}^2 \Lambda^{D-6}} \tag{3.51}$$

In the thin-brane limit ($\Lambda \to \infty$), we have $p_c \to 0$ and the infinite towers of massive modes and tachyons turns into continuous spectra. The form of the propagator in this limit is easily deduced from eq. (3.38). For momenta away from the critical scale ($|p| \gg p_c$), the two terms in (3.38) that give rise to the massive and tachyonic poles become vanishingly small and we are left with

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p)\tilde{T}'^{\alpha\beta} \sim -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\} \tag{3.52}$$

recovering four-dimensional Einstein gravity.

3.3.2 The Dubovsky-Rubakov model

It is interesting to note that similar results have been obtained by Dubovsky and Rubakov [29] using a slightly different model. In order to directly compare our results with theirs, we shall assume that the spread function (denoted by $f^2(y)$ in [29]) is given by eq. (3.7). Then the Einstein field equations proposed in [29] can be written as

$$\mathcal{F}(\Box^{(D)})G^{(D)}_{AB}(x^\mu, y^m) + \overline{M}^2 \sigma_\Lambda(y) \int d^{D-4}y' \sigma_\Lambda(y')\; G^{(4)}_{AB}(x^\mu, y'^m) = T_{AB}(x^\mu, y^m) \tag{3.53}$$

to be compared with the Einstein eq. (3.10) in our model. In eq. (3.53), the four-dimensional Einstein tensor only has brane worldvolume components (i.e., $G^{(4)}_{aB} = 0$) and the form-factor $\mathcal{F} \approx M^{D-2}$ at low energies. Also, the matter source on the brane will be assumed to have only space-time components $T_{\mu\nu}$ and a spread function same as that of the brane,

$$T_{\mu\nu}(x,y) = T_{\mu\nu}(x)\sigma_\Lambda(y) \tag{3.54}$$

where $T_{\mu\nu}(x)$ is conserved in the four-dimensional sense (*cf.* eq. (3.22) in our model). The inverse width Λ of the spread function is assumed to be $\Lambda \sim M$ in [29] to be contrasted with our model in which $\Lambda \sim \overline{M}$, since it coincides with the inverse width of the brane [9].

Working as in section 3.1, we linearize the Einstein equations and obtain the graviton propagator in the form

$$\tilde{h}_{\mu\nu}(p,y)\tilde{T}'^{\mu\nu} = \frac{2}{\mathcal{C}}\left\{\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{3}\tilde{T}^\mu_\mu \tilde{T}'^\lambda_\lambda\right\}\mathcal{G}_1(p,y) - \frac{1}{3\mathcal{C}_*}\tilde{T}^\mu_\mu \tilde{T}'^\lambda_\lambda \mathcal{G}_1(p,y) \tag{3.55}$$

where we multiplied by the arbitrary stress-energy tensor $T'_{\mu\nu}$ to absorb the longitudinal part which is not gauge-invariant. It is given in terms of the Green function which satisfies eq. (5.37) for $\lambda = 1$,

$$M^{D-2}\,(p^2 - \Box_N)\mathcal{G}_1(p,y) = \sigma_\Lambda(y) \tag{3.56}$$

(denoted by D_f in [29]). The denominators are

$$\mathcal{C} = 1 - \overline{M}^2 p^2 \mathcal{G}_1^{\text{Brane}} \quad , \quad \mathcal{C}_* = 1 + \overline{M}^2 p^2 \mathcal{G}_1^{\text{Brane}} \tag{3.57}$$

where $\mathcal{G}_1^{\text{Brane}}$ is the average of $\mathcal{G}_1$ over the spread function (defined as in eq. (3.37) and denoted by D_{ff} in [29]). Explicitly,

$$\mathcal{G}_1^{\text{Brane}}(p) = \frac{\kappa^2}{\overline{M}^2\Lambda^2}\, f(p/\Lambda) \quad , \quad f(z) = \frac{1}{z^2}\left[1 - (D-4)K_{(D-4)/2}(z)I_{(D-4)/2}(z)\right] \tag{3.58}$$

where we introduced the function $f(z)$ for convenience and the scale κ, which coincides with our earlier definition (3.33) in the large Λ limit,

$$\kappa^2 = (D-4)\frac{\Lambda^{D-4}\overline{M}^2}{\omega_{D-4}M^{D-2}} \tag{3.59}$$

The poles of the propagator (3.55) are the zeros of $\mathcal{C}$ and $\mathcal{C}_*$. They can easily be seen to correspond to small z, therefore we may approximate $\mathcal{C} \approx 1 - \kappa^2 f(0)p^2/\Lambda^2$, whose root is

$$m^2 \approx \frac{\Lambda^2}{\kappa^2 f(0)} \sim \frac{M^{D-2}}{\overline{M}^2\Lambda^{D-6}} \quad , \tag{3.60}$$

which is a massive pole. Similarly, the root of C_* is a tachyonic pole

$$m_*^2 \approx -m^2 \sim -\frac{M^{D-2}}{\overline{M}^2\Lambda^{D-6}} \tag{3.61}$$

Notice that the mass scale is similar to the mass scale of the poles in our model (3.51), although in this model only one pair of poles is obtained instead of the infinite tower we found in our model. This scale matches the one found in [29] if we set $\Lambda \sim M$, in which case $m \sim M^2/\overline{M}$.

3.4 Momentum Dependence of the Graviton Propagator

Having understood the large Λ limit, we now turn to a study of the momentum dependence of the graviton propagator keeping Λ finite. By introducing the width $1/\Lambda$, we have added a scale to the theory in addition to the mass scales M and $\overline{M}$. It follows from the explicit form of the propagator that the relevant scales are Λ and Λ/k, where $k \sim k_\lambda \sim \kappa$ is a dimensionless parameter given by (3.33) or (3.29). Phenomenologically, one expects $\Lambda \sim \overline{M}$ and $M \ll \overline{M}$. So we shall restrict attention to momenta that are well below the scale Λ ($p \ll \Lambda$). This range is divided by the scale given by eq. (3.51) into a small momentum ($p \ll p_c$) and a large momentum ($p \gg p_c$) regime. Qualitatively, one expects four-dimensional behavior of the graviton propagator for large momenta and D-dimensional behavior for small momenta [9]. We wish to study this behavior quantitatively.

For small momentum, $p \ll p_c$, we have

$$\mathcal{G}_0(p,y) \approx \mathcal{G}_\lambda(p,y) \tag{3.62}$$

as can easily be verified from eqs. (3.27) and (3.31) in the bulk and eqs. (3.26) and (3.30) on the brane. The resulting tensor structure of the graviton propagator (3.35) is

$$\tilde{h}_{\alpha\beta}(p;y)\tilde{T}'^{\alpha\beta} \simeq -\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{(D-2)}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\}\mathcal{G}_0(p,y) \tag{3.63}$$

In the bulk, we deduce from (3.31),

$$\mathcal{G}_0(p;y) \sim \left(\frac{1}{py}\right)^{(D-6)/2} K_{(D-6)/2}(py) \tag{3.64}$$

which is the propagator for a D-dimensional scalar field. Therefore, the graviton behaves as a D-dimensional field in both its momentum dependence and its tensor structure in the bulk.

On the brane, after averaging over its transverse width, eq. (3.63) yields in the regime $p \ll p_c$

$$\begin{aligned}\tilde{h}^{\mathrm{Brane}}_{\alpha\beta}(p)\tilde{T}'^{\alpha\beta} \;\; &\sim \;\; -\frac{1}{\overline{M}^2\, p^2}\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{(D-2)}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\} \\ &\times \left[1 - \frac{1}{\mathcal{A}_0\,\Gamma\left(\frac{D-4}{2}\right)}\left(\frac{\kappa p}{2\Lambda}\right)^{(D-6)/2} K_{(D-4)/2}(p/\Lambda)\right]\end{aligned} \tag{3.65}$$

where we used eqs. (3.30) and (3.37). It is easy to see that the $1/p^2$ pole vanishes. The first non-analytic term can be found from the expansion for small argument

$$K_\nu(z) = \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}(1+\ldots) + \frac{(-)^{\nu+1}}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}\ln\left(\frac{z}{2}\right) + \ldots \tag{3.66}$$

for integer ν, where the dots represent higher-order and analytic terms. Applying this to eq. (3.65), we obtain

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p)\tilde{T}'^{\alpha\beta} \sim -\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{(D-2)}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\}(-)^{(D-4)/2}\left(\frac{p}{\Lambda}\right)^{D-6}\ln(p/\Lambda) \tag{3.67}$$

exhibiting D-dimensional behavior. Similar conclusions may be drawn for the trace $\tilde{h}^{\alpha}_{\alpha}$ in the small momentum regime ($p \ll p_c$).

In the large momentum regime ($\Lambda \gg p \gg p_c$), the results are similar to those in the large Λ limit, which we discussed in the previous section. In this regime, the scalar Green functions are related by

$$\mathcal{G}_0(p,y) \approx (\lambda - 1)\mathcal{G}_\lambda(p,y) \tag{3.68}$$

to be contrasted with the relation (3.62) in the regime $p \ll p_c$. Thus the tensor structure of the graviton propagator (3.35) becomes

$$\tilde{h}_{\alpha\beta}(p,y)\tilde{T}'^{\alpha\beta} \sim -\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\}\mathcal{G}_0(p,y) \tag{3.69}$$

exhibiting four-dimensional behavior. Inside the brane, we deduce from (3.30)

$$\tilde{h}^{\text{Brane}}_{\alpha\beta}(p)\tilde{T}'^{\alpha\beta} = -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\} + \mathcal{O}\left((p_c/p)^2\right) \tag{3.70}$$

exhibiting the distance dependence of Newtonian gravity with the tensor structure of four-dimensional Einstein gravity. This is in agreement with our earlier conclusion (3.52) in the large Λ limit. In the bulk ($y > 1/\Lambda$), we deduce from (3.31)

$$\tilde{h}_{\alpha\beta}(p,y)\tilde{T}'^{\alpha\beta} \sim \frac{1}{\overline{M}^2 p^2} \left(\frac{p}{\Lambda}\right)^{1/2} \left(\frac{1}{y\Lambda}\right)^{(D-6)/2} e^{p/\Lambda} \left\{\tilde{T}_{\alpha\beta}\tilde{T}'^{\alpha\beta} - \frac{1}{2}\tilde{T}^{\alpha}_{\alpha}\tilde{T}'^{\beta}_{\beta}\right\} K_{(D-6)/2}(py) \tag{3.71}$$

Therefore, the propagator vanishes in the thin brane limit ($\Lambda \to \infty$). These results for the momentum dependence of the tensor structure of the graviton propagator are in agreement with the qualitative suggestions [9].

To recap the results of the present chapter, we've generalized the DGP model by allowing the 3-brane to have a finite thickness extending into the bulk-space. We solved the field equations and obtained the value of the graviton propagator both on the brane and in the bulk. We found that the solution for the graviton propagator contains an infinite number of massive poles and tachyonic ghosts for the 3-brane of finite thickness in this simplest setup. By allowing for this more general case of a finite brane thickness, we've effectively introduced a new scale Λ into the theory which gives rise to a critical 4D momentum p_c. In the limit that the 3-brane becomes "thin", the tensor structure of the graviton propagator reduces to that of the DGP model [9] which was presented in the previous chapter.

The graviton propagator for the 3-brane of finite thickness experiences a 4D worldvolume momentum dependence; this 4D momentum dependence corresponds to a 4D distance dependence when the inverse Fourier transform is performed. In the large 4D momentum limit, $p \gg p_c$, we recover a solution which has the tensor structure of a 4D theory. Conversely, in the small 4D momentum limit, $p \ll p_c$, we obtain a solution with a D-dimensional tensor structure.

Chapter 4

Constrained Perturbative Expansion of the DGP Model

This chapter contains a slightly revised version of a paper by the same name published in the journal *Physics Letters B* in 2005 by Chad Middleton and George Siopsis:

C. Middleton and G. Siopsis, Constrained Perturbative Expansion of the DGP Model. *Phys. Lett. B*, Vol.613 (2005) pps. 189-196 [48].

In the previous chapter, we generalized the DGP model by allowing the 3-brane to have a finite thickness into the transverse bulk-space. In this chapter, we address the issue of the van Dam-Veltman-Zacharov (vDVZ) discontinuity [30, 31] which arises in the unique case of ($D = 5$) dimensions. Our discussion follows closely to that of the Pauli-Fierz model of massive gravity which also suffers from the vDVZ discontinuity at linear order. This chapter is organized as follows. In section 4.1, we add an additional linear action contribution to the 4D Einstein-Hilbert (EH) action in order to arrive at a possible candidate for a 4D theory of massive gravity. This additional action contribution exhausts all possible linear combinations of the metric perturbations which will give rise to a local, massive propagator which is written in terms of two free parameters. After obtaining the solution for the metric perturbations for this general massive graviton theory, we explore the parameter space and find that the free parameters are severely constrained when one insists on a ghost-free, tachyon-free theory of 4D massive gravity; this brings us to the 4D model of Pauli-Fierz (PF) [42]. We compare the PF massive graviton solution to that of the EH massless theory and show that one does not recover the expected 4D Einstein solution in the limit of vanishing graviton mass of PF model; this is the vDVZ discontinuity. In section 4.2, we examine the vDVZ discontinuity of the 5D DGP model. We introduce a generalized Constrained Perturbative Expansion, which amounts to additional brane and bulk action contributions written in terms of free parameters, to cure the breakdown of the perturbative solution at linear order. By subjecting the DGP model to these general regulating conditions, we find a solution for the metric perturbations which is written in terms of these free parameters. We show that this solution has the expected 4D momentum-dependent crossover behavior and is found to be independent of the brane and bulk parameters in the near and far regimes. In the near regime, the solution reduces to that of EH action with a 4D tensor structure and distance dependence; in the far regime we find that the solution is that of a purely 5D bulk theory having a 5D tensor structure and distance dependence. We then explore this parameter space and identify the regions which give rise to non-tachyonic type resonances and hence a stable vacuum state.

4.1 The vDVZ Discontinuity of Massive Gravity

We start by presenting a possible candidate for a general theory of 4D massive gravity. In addition to the fully covariant, 4D Einstein-Hilbert action, we add an additional linearized massive contribution of the form

$$S_M = \overline{M}^2 \int d^4x \sqrt{-g} \, \left(\mathcal{R}^{(4)} + \frac{\lambda m_g^2}{4} [h_{\mu\nu} h^{\mu\nu} - \xi h^2] \right) + S_M \tag{4.1}$$

Here we have written the action in terms of free paramenters λ, ξ; this additional massive action allows for all possible combinations of the metric perturbations which will give rise to additional local field contributions to the field equations at linear order. The Pauli-Fierz action [42] of a graviton mass m_g in four spacetime dimensions is described by the action when $\lambda = 1, \xi = 1$ which, consequently, is the only allowable form which is free of tachyonic and ghost-like states, as we'll show below.

Upon varying the action (4.1), we arrive at the following linearized field equations

$$\begin{aligned} T_{\mu\nu} &= \overline{M}^2 \left[\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \Box_4 h_{\mu\nu} - \partial_\mu \partial_\nu h^\alpha_\alpha - \eta_{\mu\nu}(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box_4 h^\alpha_\alpha)\right] \\ &- \lambda m_g^2 \, \overline{M}^2 \, (h_{\mu\nu} - \xi \eta_{\mu\nu} h) \end{aligned} \tag{4.2}$$

Fourier transforming (4.2), we arrive at an equation of the form

$$\begin{aligned} \tilde{T}_{\mu\nu} &= \overline{M}^2 \left[p_\mu p^\alpha \tilde{h}_{\alpha\nu} + p_\nu p^\alpha \tilde{h}_{\alpha\mu} - p^2 \tilde{h}_{\mu\nu} - p_\mu p_\nu \tilde{h}^\alpha_\alpha - \eta_{\mu\nu}(p^\alpha p^\beta \tilde{h}_{\alpha\beta} - p^2 \tilde{h}^\beta_\beta)\right] \\ &+ \lambda m_g^2 \, \overline{M}^2 \, (\tilde{h}_{\mu\nu} - \xi \eta_{\mu\nu} \tilde{h}) \end{aligned} \tag{4.3}$$

where $p^2 = -p_\alpha^2 \equiv -p_\alpha p^\alpha$ is the 4D Euclidean momentum. As has been the notation throughout this work, quantities with tildes correspond to the Fourier transforms.

Dotting (4.3) with p^μ, we obtain the following relation

$$\lambda m_g^2 (p^\mu \tilde{h}_{\mu\nu} - \xi p_\nu \tilde{h}) = 0 \tag{4.4}$$

which is an additional constraint equation not present in the 4D Einstein-Hilbert theory ($\lambda = 0$). As is obvious from the above equation (4.4), gauge invariance of the linearized massive theory is explicitly broken by the addition of the massive action contribution (4.1). This is in contrast to the generally covariant 4D Einstein-Hilbert action which gives rise to the massless field equations in which a choice of gauge must be incorporated into the theory to break the manifest gauge invariance. Eq.(4.4) acts effectively as a gauge fixing term in the massive theory and remains in the limit of vanishing graviton mass ($m_g \to 0$). As will become obvious later in this discussion, the choice of free parameter $\xi = 1$ amounts effectively to a poor gauge choice for the 4D EH theory and will not allow for a smooth transition to 4D Einstein gravity in the massless limit.

Taking the trace and using (4.4), we arrive at an equation for the trace of the metric perturbations given by

$$\tilde{T} = \overline{M}^2 \left[2(1-\xi)p^2 - \lambda(1-4\xi)m_g^2\right] \tilde{h}_\alpha^\alpha \tag{4.5}$$

which can be inverted to yield the solution for the scalar propagator

$$\tilde{h}_\alpha^\alpha(p) = \frac{\tilde{T}}{\overline{M}^2 \left[2(1-\xi)p^2 - \lambda(1-4\xi)m_g^2\right]} \tag{4.6}$$

Now that we've arrived at the solution for the scalar propagator, we proceed to solve (4.3). After some algebra, the solution for the metric perturbations is given by

$$\tilde{h}_{\mu\nu}(p) = -\frac{1}{\overline{M}^2(p^2 + \lambda m_g^2)} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{2}(\eta_{\mu\nu} C_1 - \frac{p_\mu p_\nu}{p_\alpha^2} C_2)\tilde{T} \right\} \tag{4.7}$$

where the functions $C_1,\ C_2$ are given by

$$\begin{aligned} C_1 &= \frac{[2(1-\xi)p^2 + 2\lambda\xi m_g^2]}{[2(1-\xi)p^2 - \lambda(1-4\xi)m_g^2]} \\ C_2 &= \frac{2(1-2\xi)p^2}{[2(1-\xi)p^2 - \lambda(1-4\xi)m_g^2]} \end{aligned} \tag{4.8}$$

To study the pole structure, we decompose the metric perturbations into the following form

$$\tilde{h}_{\mu\nu}(p) = - \frac{1}{\overline{M}^2(p^2+\lambda m_g^2)}\left\{\tilde{T}_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\tilde{T}\right\} + \frac{\tilde{T}}{6\overline{M}^2(p^2+\beta\lambda m_g^2)}\left(\eta_{\mu\nu} + 2\beta\frac{p_\mu p_\nu}{p^2}\right) \tag{4.9}$$

where we've introduced the parameter β which is defined by the relation

$$\beta = \frac{(4\xi - 1)}{2(1-\xi)} \tag{4.10}$$

As is obvious from (4.9), we see that the solution for the metric perturbations have two poles when $p^2 = -\lambda m_g^2$ and $p^2 = -\beta\lambda m_g^2$. These poles give rise to massive propagators when $\lambda,\ \beta > 0$ and tachyonic-type propagators when $\lambda,\ \beta < 0$ which signals an unstable vacuum state of the theory.

So far we have treated λ, ξ as free parameters. At this point we must constrain the values of the parameters in order to ensure a well-defined theory with a stable vacuum: one that is free of tachyons and ghosts.

Examining the first pole of the metric perturbations in (4.9), we find a non-tachyonic pole when $\lambda > 0$. It follows that the metric perturbations behave as

$$\tilde{h}_{\mu\nu}(p) \sim -\frac{1}{\overline{M}^2(p^2+\lambda m_g^2)}\left\{\tilde{T}_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\tilde{T}'\right\} \tag{4.11}$$

near the massive pole $p^2 = -\lambda m_g^2$. The minus sign in front of the residue signals a well-defined amplitude.

We now proceed by examing the second pole of the metric perturbations. With $\lambda > 0$ now required from the previous analysis of the first pole, we see that a second constraint must be imposed on the free parameters of the form

$$1/4 \leq \xi \leq 1 \tag{4.12}$$

which again ensures a non-tachyonic type pole structure. Again examining the behavior near the second pole $p^2 = -\beta\lambda m^2$, we obtain

$$\tilde{h}_{\mu\nu}(p) \sim \frac{1}{6\overline{M}^2(p^2 + \beta\lambda m_g^2)} \left(\eta_{\mu\nu} + 2\beta \frac{p_\mu p_\nu}{p^2} \right) \tilde{T} \tag{4.13}$$

The plus sign of the residue of the non-tachyonic pole implies a ghost-like state and signals an inconsistency of the theory. The only value of the parameter ξ which doesn't introduce ghosts into the theory is when $\xi = 1$ in which case the mass diverges and the ghost decouples from the theory. For this completely constrained value of the parameter ξ, the solution for the metric perturbations take the form

$$\tilde{h}^{\mathrm{PF}}_{\mu\nu}(p)|_{\xi=1} = -\frac{1}{\overline{M}^2(p^2 + \lambda m_g^2)} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{3} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{\lambda m_g^2} \right) \tilde{T} \right\} \tag{4.14}$$

This solution corresponds to that of the Pauli-Fierz form [42] of massive gravity when $\lambda = 1$. It should be emphasized that $\xi = 1,\ \lambda > 0$ are the only allowable values of the free parameters that yield a purely 4D local theory containing a massive graviton propagator and are free of tachyonic-type resonances and ghost-like states.

The above results are illustrated by the two-dimensional plot of the (ξ, λ) parameter space in Figure 4.1.

This solution (4.14) should be compared with the solution of the linearized Einstein equations in the harmonic gauge, which are given by

$$\tilde{h}^{\mathrm{4D}}_{\mu\nu}(p) = -\frac{1}{\overline{M}^2 p^2} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu}\tilde{T} \right\} \tag{4.15}$$

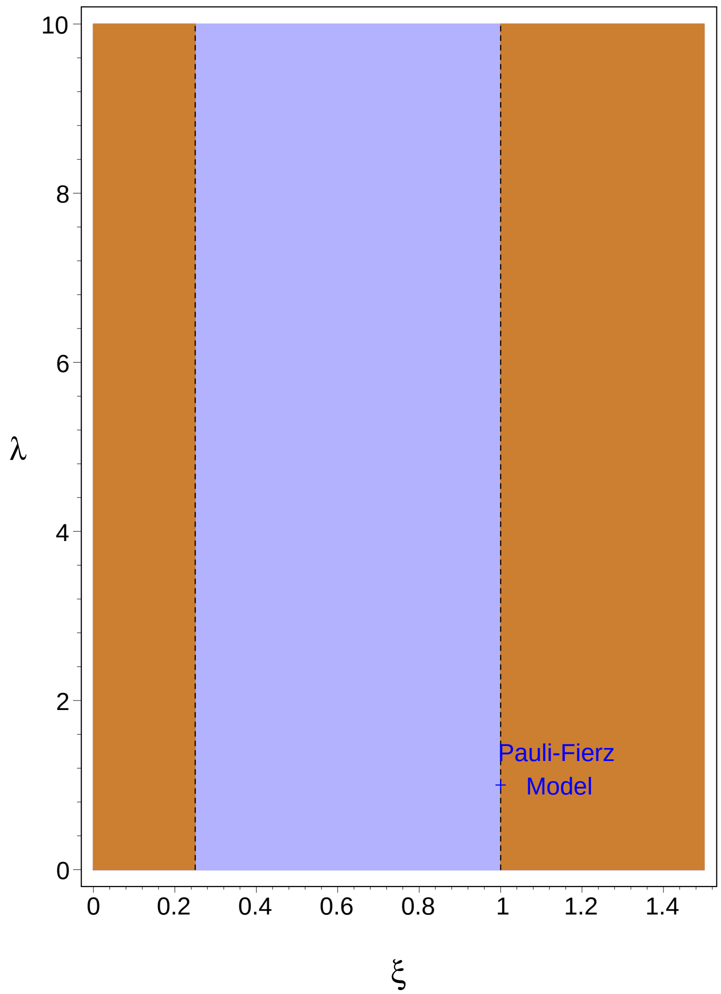

Figure 4.1: The two-dimensional (ξ, λ) parameter space of linearized massive gravity. Within the strip $1/4 < \xi < 1$, $\lambda > 0$, the theory is free of tachyonic-type poles; outside, we have tachyons (instability). The line $\xi = 1$, $\lambda > 0$ corresponds to a ghost-free and tachyon-free linear massive gravity model. The Pauli-Fierz model of massive gravity [42] is represented by the point $\xi = 1$, $\lambda = 1$.

The phenomenological differences of the massive (4.14) and the massless (4.15) cases are usually summarized by quoting the discrepancy in the prediction for the bending of light by the Sun. To see how this conflict emerges, note that the stress-energy tensor for light is traceless ($T = 0$), therefore the two expressions for the graviton agree as $m_g \to 0$, provided we set the coupling constants ($\overline{M}$) equal to each other in the two cases. However, this demand leads to a disagreement in the prediction of the strength of the gravitational force (Newton's Law). Indeed, if we couple the graviton to a conserved stress-energy tensor $T'^{\mu\nu}$, we obtain from (4.14) and (4.15), respectively,

$$\begin{aligned} \tilde{h}^{\rm PF}_{\mu\nu}(p)\tilde{T}'^{\mu\nu} &= -\frac{1}{\overline{M}^2(p^2+m_g^2)}\left\{\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{3}\tilde{T}\tilde{T}'\right\} \\ \tilde{h}^{\rm 4D}_{\mu\nu}(p)\tilde{T}'^{\mu\nu} &= -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\mu\nu}T'^{\mu\nu} - \frac{1}{2}\tilde{T}\tilde{T}'\right\} \end{aligned} \qquad (4.16)$$

To examine the behavior of the massive and massless solutions for the metric perturbations explicitly, we choose the stress-energy tensors to represent static point sources of masses m_1, m_2. By taking inverse Fourier transforms, we obtain the gravitational potentials given by the equation

$$V(r) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}}\tilde{h}_{\mu\nu}(p)\tilde{T}'^{\mu\nu} \qquad (4.17)$$

where the only non-vanishing components of the stress-energy tensor are $\tilde{T}_{00}$, $\tilde{T}'^{00}$. Plugging the values for the massive and massless graviton propagator (4.16) into eq. (4.17) gives the Newtonian and Yukawa potentials

$$V^{\rm PF}(r) = \frac{4}{3}\frac{1}{8\pi\overline{M}^2}\frac{m_1 m_2}{r}\, e^{-m_g r}$$

$$V^{4\mathrm{D}}(r) \;=\; \frac{1}{8\pi\overline{M}^2}\frac{m_1 m_2}{r} \tag{4.18}$$

which disagree with each other even in the limit $m_g \to 0$ by a factor of $4/3$. We can make them agree with each other if we demand that $\overline{M}^2$ in the massive case be 4/3 of $\overline{M}^2$ in the massless case, but then the two results would disagree in their predictions on the bending of light by the Sun. Hence the discontinuity as $m_g \to 0$ seems to have inescapable physical consequences and is commonly referred to as the van Dam-Veltmann-Zacharov (vDVZ) discontinuity of massive gravity in the literature.

To see the incompatibility of the massless limit of the Pauli-Fierz model of massive gravity with 4D Einstein gravity from the perspective of the an effective choice of gauge, it is instructive to examine the solution for the metric perturbations of the 4D Einstein-Hilbert action (Setting $m_g = 0$ in the action of (4.1)). The linearized Einstein field equations are

$$T_{\mu\nu} = \overline{M}^2\left[\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \Box_4 h_{\mu\nu} - \partial_\mu\partial_\nu h^\alpha_\alpha - \eta_{\mu\nu}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box_4 h^\alpha_\alpha)\right] \tag{4.19}$$

These equations emerge from a generally covariant theory and contain a gauge freedom. In order to obtain the solution for the metric perturbations we must fix the gauge. We make a gauge choice

$$\partial^\mu h_{\mu\nu} - \xi\partial_\nu h = 0 \tag{4.20}$$

in terms of the free parameter ξ. This gauge choice, as was previously mentioned, is equivalently (4.4) of the massive theory when the Fourier transform of (4.20) is taken.

Using this choice of gauge and taking the Fourier transforms, the solution to

(4.19) for the metric and scalar perturbations are given by

$$\begin{aligned}\tilde{h}^{4\text{D}}_{\mu\nu}(p) &= -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\mu\nu} - \frac{1}{2}\left(\eta_{\mu\nu} - \frac{(1-2\xi)}{(1-\xi)}\frac{p_\mu p_\nu}{p^2}\right)\tilde{T}\right\} \\ \tilde{h}^{\alpha\ 4\text{D}}_{\alpha}(p) &= \frac{1}{2(1-\xi)}\cdot\frac{\tilde{T}}{\overline{M}^2 p^2}\end{aligned} \tag{4.21}$$

The choice of free parameter $\xi = 1/2$ corresponds to the harmonic gauge under which the $p_\mu p_\nu$ term vanishes and we obtain the solution for the metric perturbations given by (4.15). Notice that when the choice of gauge parameter $\xi = 1$ is made, the scalar and $p_\mu p_\nu$ terms diverge and we get non-sensible solutions for the metric and scalar perturbations. Therefore, one can conclude that for the parameter choice $\xi = 1$, which in the theory of massive gravity is the only allowable choice which gives rise to a solution which has a stable vacuum state (no tachyons or ghosts), we obtain a *linear* solution which does not transition over to that of 4D massless gravity in the zero graviton mass limit.

It was pointed out by Vainshtein [32] that the above conclusion was reached in the linear approximation to the full field equations of gravity. Thus, they have a limited domain of validity. To find this domain, one ought to calculate the next-order corrections to make sure they are small. A calculation based on the Schwarzschild solution reveals that the correction to the gravitational potential due to a point source of mass m is $\mathcal{O}(1/m_g^4)$ and therefore singular in the limit $m_g \to 0$. One obtains [33]

$$V^{\text{PF}}(r) = \frac{4}{3}\,\frac{1}{8\pi\overline{M}^2}\,\frac{m_1 m_2}{r}\left(1 + \mathcal{O}\left(\frac{m}{\overline{M}^2 m_g^4 r^5}\right)\right) \tag{4.22}$$

Thus the linear approximation is only reliable in the regime

$$r \gg R \quad , \quad R = \left(\frac{m}{\overline{M}^2 m_g^4}\right)^{1/5} \tag{4.23}$$

It was also shown in [32] (see the next chapter for a detailed discussion) that for $r_m \ll r \ll R$, where r_m is the Schwarzschild radius, the potential is

$$V^{\mathrm{PF}}(r) = \frac{1}{8\pi \overline{M}^2} \frac{m_1 m_2}{r} \left(1 + \mathcal{O}\left(\frac{m_g^2 \sqrt{m r^3}}{\overline{M}}\right)\right) \tag{4.24}$$

with m_g sufficiently small ($m_g R \lesssim 1$). In this regime, the massive potential has a smooth limit as $m_g \to 0$ which coincides with the result from Einstein's theory, $V^{4\mathrm{D}}(r)$ (eq. (4.18)). Thus, there is no discontinuity when the correct expansion for a physical quantity is performed. The extra degrees of freedom of the spin-2 field $h^{\mathrm{PF}}_{\alpha\beta}$ decouple in the massless limit $m_g \to 0$.

4.2 The Constrained Perturbative Expansion

In the previous section, we introduced a linearized theory of massive gravity initially written in terms of free parameters. We obtained the solution for the metric perturbations and examined the pole structure which severely constrained the values of the parameters when one insists on a local theory which is free of tachyonic and ghost-like states. Here, we present a generalized procedure of [46]. As was done in the previous section, we introduce a two-parameter family of gauge-fixing type terms on the 4D worldvolume. In addition to this brane contribution, we also include a gauge-fixing type term in the bulk in terms of arbitrary bulk parameters. In the decoupling limit and in the absence of the brane, these additional action contributions amount to ordinary gauge-fixing terms. We then proceed to explore the physical effects of these parameters away from the two extremal limits (decoupling and absence of a brane). We find that the graviton propagator in general has a well-defined decoupling limit implying the absence of a vDVZ discontinuity. The graviton propagator exhibits the expected crossover behavior and is found to be free of tachyonic asymptotic states.

The DGP solution (2.51), [8] of a delta-function type brane in ($D = 5$) corresponds to a set of measure zero in our parameter space.

The DGP model describes a 3-brane on the boundary of a five-dimensional bulk-space Σ. The action is

$$S_{DGP} = M^3 \int_{\Sigma} d^4x dy \sqrt{-G}\, \mathcal{R}^{(5)} + \overline{M}^2 \int_{\partial\Sigma} d^4x \sqrt{-g}\, \mathcal{R}^{(4)} \tag{4.25}$$

where $\mathcal{R}^{(5)}, (\mathcal{R}^{(4)})$ is the five- (four-) dimensional Ricci scalar. We adopt the standard conventions $\eta_{AB} = \text{diag}[+----]$; $A, B = 0, \ldots, 3, y$; $\mu\nu = 0, \ldots, 3$; $i, j = 1, 2, 3$.

Upon varying (4.25), one arrives at the DGP field equations, which are

$$M^3 G^{(5)}_{AB} + \overline{M}^2 G^{(4)}_{\mu\nu} \delta^\mu_A \delta^\nu_B \delta(y) = T_{\mu\nu} \delta^\mu_A \delta^\nu_B \delta(y) \tag{4.26}$$

with the linearized solution given by

$$\begin{aligned} \tilde{h}_{\mu\nu}(p, y) &= -\frac{1}{\overline{M}^2 (p^2 + 2m_b p)} \left\{ \tilde{T}_{\mu\nu} - \frac{1}{3} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{2m_b p} \right) \tilde{T} \right\} e^{-py} \\ \tilde{h}^\alpha_\alpha &= \frac{\tilde{T}}{6\overline{M}^2 m_b p} e^{-py} = \tilde{h}^y_y \end{aligned} \tag{4.27}$$

which is written in terms of the $4D$ Euclidean momentum and graviton mass

$$\begin{aligned} p^2 &= -p^\mu p_\mu = -p_0^2 + p_i^2 = p_4^2 + p_i^2 \\ m_b &= M^3 / \overline{M}^2 \end{aligned} \tag{4.28}$$

The solution bears a striking resemblance to that of PF massive gravity presented in the previous section where the factor of 1/3 instead of the Einstein factor of 1/2 signals the existence of a vDVZ discontinuity. In the decoupling limit ($m_b \to 0$), 4D Einstein gravity is not recovered and we do not obtain sensible dynamics for the longitudinal term with tensor structure of the form $p_\mu p_\nu$. Although the $p_\mu p_\nu$ term does not contribute at linear level, it does enter nonlinear diagrams.

Generalizing [46], we define a *Constrained DGP Action* of the form

$$S_{cDGP} = S_{DGP} + S^{(5)} + S^{(4)} \tag{4.29}$$

where S_{DGP} is the DGP action given by (4.25) and $S^{(4)}$ and $S^{(5)}$ are gauge-fixing terms in the decoupling limit ($m_b \to 0$) and absence of brane ($m_b \to \infty$), respectively. Away from these two limits ($M, \overline{M} \neq 0$), these additional terms no longer simply fix the gauge; they alter the boundary conditions.

We start in the bulk by defining $S^{(5)}$ as follows

$$S^{(5)} = M^3 \int_\Sigma d^4x dy \, \sqrt{-G} \left[\frac{B_5^2}{2\gamma} + \frac{B_\mu^2}{2\alpha} \right] \tag{4.30}$$

with

$$\begin{aligned} B_\mu &\equiv \partial_\mu h^y_y + a\partial_\mu h^\alpha_\alpha - b\partial^\alpha h_{\alpha\mu} \\ B_y &\equiv \partial^\mu h_{\mu y} \end{aligned} \tag{4.31}$$

where α, γ, a, b are arbitrary parameters on which no bulk physical quantities should depend. In the absence of the brane, eq. (4.30) amounts to standard gauge-fixing conditions. In general, the $\alpha, \gamma \to 0$ limit should be taken at the end of the calculation to ensure that

$$\begin{aligned} B_\mu &\to 0 \\ B_y &\to 0 \end{aligned} \tag{4.32}$$

Next, we define the gauge-fixing term $S^{(4)}$ on the brane. For a brane of finite thickness, additional terms can arise on the brane world-volume and can survive in the limit of the brane thickness tending to zero. In addition, we note that the boundary equations receive no contribution from eq. (4.30) and are invariant under the $4D$ transformations [46]

$$h_{\mu\nu}|_{y=0} \to h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu |_{y=0} \tag{4.33}$$

indicating a residual gauge freedom. With the above in mind, we choose an additional brane action contribution

$$S^{(4)} = \lambda\, \overline{M}^2 \int_{\partial\Sigma} d^4x \sqrt{-g}\, \mathcal{B}_\nu^2 \tag{4.34}$$

where

$$\mathcal{B}_\nu \equiv \partial^\mu h_{\mu\nu} + \xi \partial_\nu h^\alpha_\alpha \tag{4.35}$$

and we assume $\lambda > 0$. These additional action contributions modify the DGP model by explicitly breaking the $4D$ and $5D$ coordinate invariance. Adopting this modified DGP model, we next obtain and solve the field equations. Varying (4.29), expanding around a flat background, and Fourier transforming, the first-order Einstein equations are as follows. In the bulk, the trace of the transverse component (55) is

$$\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2} \tilde{h}_{\alpha\beta} - \frac{1}{\alpha} \left(\tilde{h}^y_y + a \tilde{h}^\alpha_\alpha - b \frac{p^\alpha p^\beta}{p^2} \tilde{h}_{\alpha\beta} \right) = 0 \tag{4.36}$$

The mixed components ($\mu 5$) are

$$i\partial_y (p^\alpha \tilde{h}_{\alpha\mu} - p_\mu \tilde{h}^\alpha_\alpha) + p^2 \left(\tilde{h}_{\mu y} - \frac{1}{p^2} p_\mu p^\alpha \tilde{h}_{\alpha y} \right) + \frac{1}{\gamma} p_\mu p^\alpha \tilde{h}_{\alpha y} = 0 \tag{4.37}$$

and the components parallel to the brane ($\mu\nu$) are

$$\mathcal{G}^{(5)}_{\mu\nu} = 0 \tag{4.38}$$

where

$$
\begin{aligned}
\mathcal{G}^{(5)}_{\mu\nu} &= (p^2 - \partial_y \partial^y)(\tilde{h}_{\mu\nu} - \eta_{\mu\nu}\tilde{h}^{\alpha}_{\alpha}) - p_\mu p^\alpha \tilde{h}_{\alpha\nu} - p_\nu p^\alpha \tilde{h}_{\alpha\mu} + p_\mu p_\nu (\tilde{h}^{\alpha}_{\alpha} + \tilde{h}^{y}_{y}) \\
&- \eta_{\mu\nu}(p^2 \tilde{h}^y_y - p^\alpha p^\beta \tilde{h}_{\alpha\beta}) \\
&+ \frac{1}{\alpha}\Big[-b(p_\mu p_\nu \tilde{h}^y_y + a p_\mu p_\nu \tilde{h}^\alpha_\alpha - b p_\nu p^\alpha \tilde{h}_{\alpha\mu}) + a\eta_{\mu\nu}(p^2\tilde{h}^y_y + a p^2 \tilde{h}^\alpha_\alpha - b p^\alpha p^\beta \tilde{h}_{\alpha\beta})\Big] \\
&+ \partial^y (p_\nu \tilde{h}_{\mu y} + p_\mu \tilde{h}_{\nu y} - 2\eta_{\mu\nu} p^\alpha \tilde{h}_{\alpha y}) \qquad (4.39)
\end{aligned}
$$

From eqs. (4.36) and (4.37), we obtain

$$
\begin{aligned}
\tilde{h}_{\mu y} &= i\gamma \frac{p_\mu}{p^2}\, \partial_y \left(\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2}\tilde{h}_{\alpha\beta}\right) \\
\tilde{h}^y_y &= b\frac{p^\alpha p^\beta}{p^2}\tilde{h}_{\alpha\beta} - a\tilde{h}^\alpha_\alpha + \alpha\left(\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2}\tilde{h}_{\alpha\beta}\right) \qquad (4.40)
\end{aligned}
$$

Plugging these expressions into (4.39) and assuming the solution is of the form

$$\tilde{h}_{AB}(p, y) = \tilde{h}_{AB}(p) e^{-py} \qquad (4.41)$$

we may write $\mathcal{G}^{(5)}_{\mu\nu}$ entirely in terms of the $4D$ metric perturbations. Dotting with the momentum, we obtain

$$p^\mu p^\nu \mathcal{G}^{(5)}_{\mu\nu} = (1 + a - b)p^2(p^2\tilde{h}^\alpha_\alpha - p^\alpha p^\beta \tilde{h}_{\alpha\beta}) \qquad (4.42)$$

implying the constraint on the parameters

$$1 + a - b = 0 \qquad (4.43)$$

The vanishing of the divergence, $p^\nu \mathcal{G}^{(5)}_{\mu\nu} = 0$, then implies

$$p^\nu \tilde{h}_{\mu\nu} = p_\mu \frac{p^\alpha p^\beta}{p^2} \tilde{h}_{\alpha\beta} \tag{4.44}$$

This is not an additional constraint on the metric. On general grounds, one may argue that $p^\nu \tilde{h}_{\mu\nu} \propto p_\mu$, hence (4.44). Using these results, we arrive at the expression

$$\mathcal{G}^{(5)}_{\mu\nu} = (\alpha - 2\gamma - 2a)(p_\mu p_\nu - \eta_{\mu\nu} p^2) \left(\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2} \tilde{h}_{\alpha\beta} \right) \tag{4.45}$$

leading to a second constraint on the parameters,

$$\alpha - 2\gamma - 2a = 0 \tag{4.46}$$

At the boundary, the Israel junction condition at $y = 0$ yields

$$\overline{M}^2 \mathcal{G}^{(4)}_{\mu\nu} = \tilde{T}_{\mu\nu} \tag{4.47}$$

where

$$\begin{aligned} \mathcal{G}^{(4)}_{\mu\nu} &= (p^2 + 2m_b p)(\tilde{h}_{\mu\nu} - \eta_{\mu\nu} \tilde{h}^\alpha_\alpha) - (1 - \lambda)(p_\mu p^\alpha \tilde{h}_{\alpha\nu} + p_\nu p^\alpha \tilde{h}_{\alpha\mu}) \\ &+ (1 + 2\lambda\xi)(p_\mu p_\nu \tilde{h}^\alpha_\alpha + \eta_{\mu\nu} p^\alpha p^\beta \tilde{h}_{\alpha\beta}) + 2\lambda\xi^2 \eta_{\mu\nu} p^2 \tilde{h}^\alpha_\alpha \\ &+ 2\gamma m_b p \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left(\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2} \tilde{h}_{\alpha\beta} \right) \end{aligned} \tag{4.48}$$

Eq. (4.47) can be solved for arbitrary parameters λ, ξ and γ. We obtain on the brane

$$\tilde{h}_{\mu\nu}(p) = -\frac{1}{\overline{M}^2 (p^2 + 2m_b p)} \left\{ \tilde{T}_{\mu\nu} - \left(\eta_{\mu\nu} \mathcal{C}_1 + \frac{p_\mu p_\nu}{p^2} \mathcal{C}_2 \right) \tilde{T} \right\} \tag{4.49}$$

where

$$
\begin{aligned}
\mathcal{C}_1 &= \frac{2m_b^2 + 2\lambda(1+\xi)(1-\xi+2\gamma(1+\xi))m_b p + \lambda(1+\xi)^2 p^2}{6m_b^2 + 4\lambda(1+\xi)(1-2\xi+3\gamma(1+\xi))m_b p + 2\lambda(1+\xi)^2 p^2} \\
\mathcal{C}_2 &= \frac{(1-2\lambda(1+\xi)(1+2\gamma(1+\xi))m_b p - \lambda(1+\xi)(1+2\xi)p^2}{6m_b^2 + 4\lambda(1+\xi)(1-2\xi+3\gamma(1+\xi))m_b p + 2\lambda(1+\xi)^2 p^2}
\end{aligned}
\tag{4.50}
$$

Notice that the $4D$ metric perturbations, when convoluted with a conserved tensor $\tilde{T}'^{\mu\nu}$,

$$
\tilde{h}_{\mu\nu}\tilde{T}'^{\mu\nu} = -\frac{1}{\overline{M}^2(p^2+2m_b p)}\left\{\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \mathcal{C}_1\tilde{T}\tilde{T}'\right\} \tag{4.51}
$$

are still dependent on the parameters λ, ξ and γ. Examining the $4D$ momentum dependence of the metric perturbations, we find in the large momentum regime ($p \gg m_b$),

$$
\tilde{h}_{\mu\nu}\tilde{T}'^{\mu\nu} \simeq -\frac{1}{\overline{M}^2 p^2}\left\{\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{2}\tilde{T}\tilde{T}'\right\} \tag{4.52}
$$

recovering 4D Einstein gravity, and in the small momentum limit ($p \ll m_b$),

$$
\tilde{h}_{\mu\nu}\tilde{T}'^{\mu\nu} \simeq -\frac{1}{2M^3 p}\left\{\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{3}\tilde{T}\tilde{T}'\right\} \tag{4.53}
$$

exhibiting 5D behavior, as expected. Notice that in both limits, the transverse components of the metric on the brane are independent of the parameters λ, ξ and γ. In the intermediate range, the propagator smoothly switches from the 4D expression (4.52) to the 5D expression (4.53) as the momentum decreases. This crossover behavior depends on the parameters λ, ξ and γ.

In the decoupling limit, $m_b \to 0$, the graviton propagator yields the standard $4D$ Einstein solution on the brane demonstrating the absence of a vDVZ discontinuity. This is the case in the entire parameter space except for a set of measure zero defined by

$$\xi = -1 \tag{4.54}$$

For this special choice, the parameters become true gauge parameters throughout the entire range of momenta. We obtain

$$\begin{aligned}
\tilde{h}_{\mu\nu}(p,y) &= -\frac{1}{\overline{M}^2(p^2+2m_b p)}\left\{\tilde{T}_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{2m_b p}\right)\tilde{T}\right\} e^{-py} \\
\tilde{h}^y_y(p,y) &= \frac{1}{6\overline{M}^2 m_b p}\,\tilde{T}\, e^{-py} \\
\tilde{h}_{\mu y}(p,y) &= 0
\end{aligned} \tag{4.55}$$

which is independent of α, γ. Also, the constraints $B_\mu = B_5 = 0$ for general α, γ showing that they represent gauge-fixing conditions. This is the solution of the standard DGP model (2.51), [8].

It should also be noted that for the particular choice of parameters $\lambda = 1,\ \xi = -1/2$, we recover the model proposed by Gabadadze [46],

$$\begin{aligned}
\tilde{h}_{\mu\nu}(p,y) &= -\frac{1}{\overline{M}^2(p^2+2m_b p)}\left\{\tilde{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\frac{(p^2+4m_b p)}{(p^2+6m_b p)}\tilde{T}\right\} e^{-py} \\
\tilde{h}^y_y &= \frac{p^\alpha p^\beta}{p^2}\tilde{h}_{\alpha\beta} + (\tfrac{1}{2}\alpha + \gamma)\left(\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2}\tilde{h}_{\alpha\beta}\right) \\
\tilde{h}_{\mu y} &= \gamma\frac{p_\mu}{p}\left(\tilde{h}^\alpha_\alpha - \frac{p^\alpha p^\beta}{p^2}\tilde{h}_{\alpha\beta}\right)
\end{aligned} \tag{4.56}$$

in the $\alpha, \gamma \to 0$ limit.

We next wish to examine the poles of the propagator. Taking the $\gamma \to 0$ limit, the transverse part of the propagator (4.51) can be written in a form explicitly revealing its pole structure,

$$\tilde{h}_{\mu\nu}\tilde{T}'^{\mu\nu} = -\frac{1}{\overline{M}^2}\left[\frac{1}{(p^2+2m_b p)}\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{3}\mathcal{C}(p)\tilde{T}\tilde{T}'\right] \tag{4.57}$$

where

$$\mathcal{C}(p) = \frac{1}{p^2 + 2m_b p} + \frac{1}{2(c_+ - c_-)}\left[\frac{c_+}{(p^2 + c_+ m_b p)} - \frac{c_-}{(p^2 + c_- m_b p)}\right] \tag{4.58}$$

The location of the poles is determined by the coefficients

$$c_\pm = c_\pm(\xi, \lambda) = \frac{1}{1+\xi}\left[1 - 2\xi \pm \sqrt{(1-2\xi)^2 - \frac{3}{\lambda}}\right] \tag{4.59}$$

For $p \gg m_b$, $\mathcal{C}(p) \approx \frac{3}{2p^2}$ and we recover the 4D expression (4.52). The poles are significant for momenta $p \lesssim m_b$. As was shown in [46], the $p = -2m_b$ pole lies on the second Riemann sheet in the Minkowski four-momentum complex plane, where $p^2 = s\,\exp(-i\pi)$, $s = p_\mu p^\mu$. This pole corresponds to a non-physical resonance and indicates an intermediate, metastable state. This can be seen from the $p = \pm\sqrt{-s}$ dependence of the propagator which indicates that the propagator is multi-valued and the complex s-plane has two sheets with a branch cut on the positive real axis. For the choice of $p = \sqrt{-s}$, we obtain a non-physical resonance and a propagator which decays with the bulk coordinate.

The other two poles are located at $p = -c_\pm m_b$ and depend on the parameters ξ and λ. In the (ξ, λ)-plane, above the curve

$$\lambda = \frac{3}{(1-2\xi)^2} \tag{4.60}$$

both poles lie on the negative real axis in the complex s-plane, since $c_\pm \in \mathbb{R}$. Moreover, $c_\pm > 0$ for $-1 < \xi < 1/2$. In this strip, the two poles are in the second Riemann sheet (corresponding to the choice $p = \sqrt{-s}$) and are thus unphysical. In the special case $\xi = -1/2$, $\lambda = 1$, the pole at $p = -c_- m_b$ coincides with the pole at $p = -2m_b$; this is the Gabadadze model [46]. As we approach the curve (4.60), the two poles merge.

Below the curve (4.60), $c_\pm$ become complex and $c_+ = c_-^*$. In this case the poles are no longer on the real axis; we obtain a resonance with a momentum independent decay width, in addition to the pole at $p = -2m_b$.

As $\xi \to -1$, the two poles $p = c_\pm m_b$ become infinite and $\mathcal{C}(p) \to (p^2+2m_b p)^{-1}$. This is a singular case; dependence on the λ parameter disappears and the propagator turns into the DGP expression (4.55) [8] which is plagued by the vDVZ discontinuity.

To the left of $\xi = -1$ (as well as for $\xi > 1/2$), both $c_\pm < 0$; therefore, the poles $p = -c_\pm m_b$ are tachyons, signaling instability of the solution. Were we to choose $p = -\sqrt{-s}$, instead, we would place these two poles on the second Riemann sheet, but then the third pole at $p = -2m_b$ would turn into a tachyon.

The above results are illustrated by the two-dimensional plot of the (ξ, λ) parameter space in Figure 4.2.

To summarize the results of this chapter, we first investigated a 4D theory of massive gravity given in terms of free parameters. After obtaining the solution for the metric perturbations, we proceeded to examine the parameter space and found that the free parameters of our theory were completely constrained in order to arrive at a theory which was free of tachyonic-type resonances and ghost-like states. We then applied a similar mechanism to generalize the constrained perturbative model of [46] and calculated the graviton propagator. The first-order contribution to the perturbative expansion depended explicitly on parameters which are gauge parameters in the bulk (in the absence of a brane) and on the brane (in the decoupling limit), respectively. These parameters determine the details of the distance-dependent, crossover behavior of the propagator and the position of the poles of the graviton propagator. At low momenta, we obtained a 5D behavior whereas at high momenta we recovered 4D gravity demonstrating the absence of a vDVZ discontinuity. In addition, we found a range of parameter values which yielded non-physical resonances corresponding to intermediate, metastable states. For a special choice of parameters (representing a set

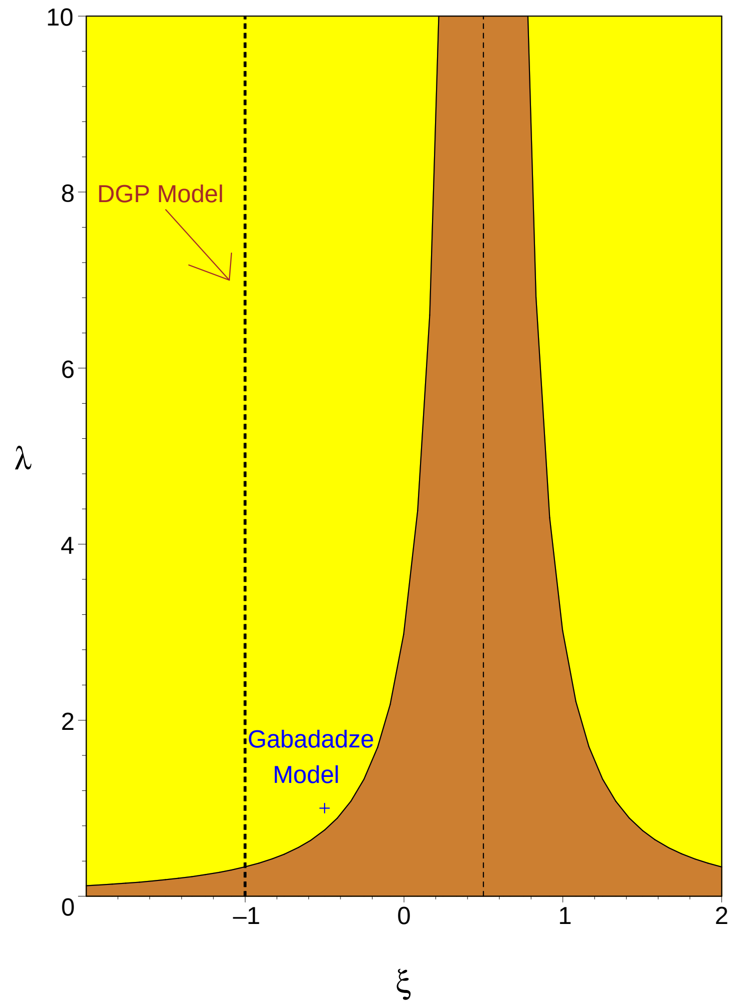

Figure 4.2: The two-dimensional (ξ, λ) parameter space of the DGP model in 5-dimensions. Above the curve (4.60), all poles of the propagator are real. Within the strip $-1 < \xi < 1/2$, only unphysical resonances appear; outside, we have tachyons (instability). Below the curve (4.60), we have one real pole and a resonance with momentum independent decay width. The DGP model [8] is represented by the line $\xi = -1$; the Gabadadze model [46] by the point $\xi = -1/2$, $\lambda = 1$.

of measure zero in the parameter space), we recovered the standard DGP model [8]. This choice represented a set of measure zero in the parameter space which is plagued by the vDVZ discontinuity.

Chapter 5

The Schwarzschild Solution in Pauli-Fierz Massive Gravity and the DGP Model

This chapter contains a lightly revised version of a paper published in the journal *Modern Physics Letters A* in 2004 by Chad Middleton and George Siopsis:

C. Middleton and G. Siopsis, The Schwarzschild Solution in the DGP Model. *Mod. Phys. Lett. A*, Vol.19 (2004) pps. 2259-2266 [37].

In the previous chapter, we witnessed the vDVZ discontinuity of massive gravity and the DGP Model in 5D. This discontinuity arises from the breakdown of the perturbative expansion and is an artifact of the linear approximation to the full non-linear field equations, thus signalling a limited domain of validity of the expansion. We showed that this breakdown can be cured by adopting a constrained perturbative expansion. Thus the theory was regulated by the inclusion of additional brane and bulk action contributions which modified the linearized field equations and allowed for a well-behaved solution.

In this chapter, we readdress the vDVZ discontinuity of the the Pauli-Fierz model and the 5D DGP Model for the case of the spherically symmetric solution of a massive point source. The chapter is organized as follows. In section 5.1 we adopt a spherically symmetric metric ansatz and obtain the full, non-linear field equations for the case of a point source. By performing a perturbative expansion, we arrive at the first-order field equations and explicitly witness the breakdown of the linearized field. We then search for a solution in the small graviton mass regime and show that by first taking the vanishing graviton mass limit and then keeping all lowest-order contributions of the metric in the field equations, which amounts to keeping *second-order* contributions in one of the fields, we arrive at such a solution. In section 5.2, we then discuss the perturbative solution to the DGP field equations in the case of a point source. By employing a spherically symmetric ansatz for the metric, in addition to off-diagonal metric contributions, and keeping all lowest-order contributions to the field equations, which again includes *second-order* terms in one of the fields, we arrive at a lowest-order approximation to the full field equations and obtain a solution. This interpolating solution is found explicitly, throughout its domain of validity (both near and far from the Schwarzschild radius), on the brane and in the bulk. We then examine the solution in the near and far regime and see that the solution reduces to that of the 4D Einstein solution in the decoupling limit.

5.1 The Absence of the vDVZ Discontinuity in Massive Gravity

In the first section of the previous chapter, we found that the most general 4D model of linearized massive gravity is severely constrained to be that of the PF model if one requires that the model be free of tachyonic and ghost-like states. After performing a perturbative expansion, we arrived at the solution which is inevitably plagued by a vDVZ discontinuity. This solution in the linear approximation has a limited range of validity; the perturbative solution breaks down in the limit of vanishing graviton mass [32].

In this section, we return to the PF model of massive gravity. We wish to obtain the spherically symmetric solution for a static point source valid in the small graviton mass regime, which transitions over to the 4D Einstein gravity solution in the massless limit. To arrive at such a solution, we will need to perform a different expansion keeping up to *second-order* terms in the fields in this small graviton mass regime.

The Pauli-Fierz field equations of massive gravity for a static point source are

$$\overline{M}^2 \left[(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) - \frac{1}{2} m_g^2 (h_{\mu\nu} - \eta_{\mu\nu} h^\alpha_\alpha) \right] = T_{\mu\nu} \tag{5.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the 4D Einstein tensor, $R = R^\alpha_\alpha$ the 4D Ricci scalar, and the stress-energy tensor for the massive point source is given by

$$T_{\mu\nu} = m \delta^0_\mu \delta^0_\nu \delta^3(\vec{x}) \tag{5.2}$$

We choose the spherically symmetric metric *ansatz*

$$ds^2 = -e^{2B(\bar{r})} dt^2 + e^{2\bar{C}(\bar{r})} d\bar{r}^2 + \bar{r}^2 e^{2A(\bar{r})} (d\theta^2 + \sin^2\theta d\phi^2) \tag{5.3}$$

which is the most general form of a spherically symmetric metric. General Relativity is unique, as compared with other physical theories, in that one simultaneously defines coordinates and the metric as a function of those coordinates. Due to this generic arbitrariness of defining coordinates, one is free to define new coordinates or to perform coordinate transformations [57]. For the case of the above metric ansatz, it is convenient to make the following coordinate transformation

$$r = \bar{r}e^{A} \tag{5.4}$$

together with the substitution

$$e^{2\bar{C}} = \frac{e^{2(C+A)}}{(1 - rA')^2} \tag{5.5}$$

where the prime denotes differentiation with respect to r. In 4D Einstein gravity ($m_g = 0$) under the above coordinatate transformation and substitution, the function $A(r)$ is completely eliminated from the field equations. One is left with the Einstein field equations written only in terms of the functions $B(r)$ and $C(r)$. This implies that the spherically symmetric metric ansatz

$$ds^2 = -e^{2B(r)}dt^2 + e^{2C(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.6}$$

is as general as (5.3) for the generally covariant theory. Alternatively stated, for the generally covariant theory one can begin by plugging the metric ansatz (5.3) into the Einstein field equations, perform the coordinate transformation and substitution given by (5.4) and (5.5) and obtain field equations which are identical to those that are obtained for the latter choice of metric given by (5.6). The vanishing of the function $A(r)$ under the coordinate transformation and substitution stated above is only true for the generally covariant Einstein equations. When the Pauli-Fierz field equations (4.1) are considered, the function $A(r)$ remains when the above transformations are performed.

We now wish to obtain the field equations in terms of the functions $A(r), B(r)$, and $C(r)$. Plugging the metric ansatz (5.3) into the full non-linear field equations and then performing the coordinate transformation and substitution given by (5.4) and (5.5), we obtain the following equations.

The tt and rr components of the full, non-linear field equations are given by the relations

$$
\begin{aligned}
e^{2(B-C)}\left[\frac{2}{r}C' - \frac{1}{r^2}(1-e^{2C})\right] &+ \frac{1}{2}m_g^2\left[\frac{e^{2(C+A)}}{(1-rA')^2} + 2e^{2A} - 3\right] = \frac{m}{\overline{M}^2}\,\delta^3(\vec{x}) \\
\left[\frac{2}{r}B' + \frac{1}{r^2}(1-e^{2C})\right] &+ \frac{1}{2}m_g^2\left[3 - e^{2B} - 2e^{2A}\right] = 0
\end{aligned} \tag{5.7}
$$

One arrives at the third unique field equation by taking the divergence of (5.1). This yields the relation

$$
-\frac{1}{2}m_g^2\left(\nabla^\mu h_{\mu\nu} - \nabla_\nu h\right) = 0 \tag{5.8}
$$

In terms of the functions in the metric ansatz, we obtain the equation

$$
\begin{aligned}
- &\; 2re^{-2(C+A)}(1-rA')^2\left[B'e^{2B} + 2A'e^{2A}\right] + 2e^{-2A}\left[e^{2B} + e^{2A} + \frac{e^{2(C+A)}}{(1-rA')^2} - 3\right] \\
+ &\; e^{-2(C+A)}(1-rA')^2\left[3 - e^{2B} - 2e^{2A}\right](2 + rB') + rB'e^{-2B}\left[\frac{e^{2(C+A)}}{(1-rA')^2} + 2e^{2A} - 3\right] \\
+ &\; r\left[3 - e^{2B} - 2e^{2A}\right]\frac{d}{dr}\left[e^{-2(C+A)}(1-rA')^2\right] = 0
\end{aligned} \tag{5.9}
$$

As was stated in the first section of the previous chapter, this equation (5.9) is responsible for the breakdown of the perturbative solution when only first-order contributions are kept. To witness this breakdown for the spherically symmetric solution for a massive point source, we expand to first-order in the fields. Under this expansion, one obtains the following linearized field equations

$$
\frac{1}{r^2}\frac{d}{dr}(rC) + \frac{1}{2}m_g^2\left[C + \frac{1}{r^2}\frac{d}{dr}(r^3A)\right] = \frac{m}{\overline{M}^2}\,\delta^3(\vec{x}) \tag{5.10}
$$

$$
\frac{2}{r^2}(C - rB') + m_g^2\,(B + 2A) = 0 \tag{5.11}
$$

and the divergence (5.9) yields the relation

$$C - \frac{1}{2}\, rB' = 0 \tag{5.12}$$

One can easily see that this additonal relation (5.12), which is absent in 4D Einstein theory, is not compatible with (5.11) in the zero graviton mass limit ($m_g \to 0$), hence the vDVZ discontinuity of the linear PF field equations. Using eqs.(5.10), (5.11), and (5.12) and performing some algebra, the equations can be decoupled and brought into the form

$$\begin{aligned} (\nabla^2 - m_g^2)B(r) &= \frac{2}{3}\frac{m}{\overline{M}^2}\, \delta^3(\vec{x}) \\ A(r) &= \frac{1}{2m_g^2 r}B'(r) - \frac{1}{2}B(r) \\ C(r) &= \frac{1}{2}\, rB'(r) \end{aligned} \tag{5.13}$$

These equations yield the following solutions

$$\begin{aligned} B(r) &= -\frac{4}{3}\left(\frac{m}{8\pi\overline{M}^2 r}\right)\, e^{-m_g r} \\ C(r) &= \frac{2}{3}\left(\frac{m}{8\pi\overline{M}^2 r}\right)(1 + m_g r)\, e^{-m_g r} \\ A(r) &= \frac{2}{3}\left(\frac{m}{8\pi\overline{M}^2 r}\right)\left[1 + \frac{1}{m_g r} + \frac{1}{m_g^2 r^2}\right]\, e^{-m_g r} \end{aligned} \tag{5.14}$$

As can be seen from the above solutions, $A(r)$ diverges in the $m_g \to 0$ limit for finite distance. One should of course demand that the functions A, B, C remain small for a valid perturbative expansion. The failure of the solutions to remain small in the massless graviton limit signals the breakdown of the perturbative expansion and defines the limited domain of validity.

After witnessing the breakdown of the perturbative solution for the spherically symmetric solution in the small graviton mass regime, we wish to now arrive at a solution which smoothly transitions into the weak-field Schwarzschild solution in the vanishing graviton mass limit. We perform an expansion which differs from that of the perturbative expansion [32] and is described as follows.

To lowest order in the small graviton mass limit, (5.10) and (5.11) reduce to

$$\begin{aligned} 2\nabla^2 D(r) &= \frac{m}{\overline{M}^2}\delta^3(\vec{x}) \\ B' - D' &= 0 \end{aligned} \tag{5.15}$$

where we wrote the equations in terms of the $D(r)$ where $C = rD'$. As has already been mentioned, the breakdown of the perturbative solution arises from the addition constraining equation (5.12). As can easily be seen from (5.12), the field $A(r)$ is absent at linear order and leads to the incompatibility of (5.12) and (5.15). To arrive at a set of compatible equations in this zero graviton mass limit, we shall keep all lowest-order contributions in the field equations. This amounts to keeping *second-order* terms in the field $A(r)$ in addition to the first-order contributions of fields $B(r)$ and $C(r)$. Performing this expansion on (5.9), we arrive at the consistant relation

$$C - \frac{1}{2}rB' + (8rAA' + \frac{7}{2}r^2A'^{\,2} + 2r^2AA'') = 0 \tag{5.16}$$

Solving (5.15) and (5.16), we obtain the solutions

$$\begin{aligned} B(r) &= -\frac{m}{8\pi\overline{M}^2 r} \\ C(r) &= \frac{m}{8\pi\overline{M}^2 r} \\ A(r) &= \sqrt{\frac{4}{13}}\left(\frac{m}{8\pi\overline{M}^2 r}\right)^{1/2} \end{aligned} \tag{5.17}$$

which yields the relation $B + C = 0$ which corresponds to the Schwarzschild solution.

This solution is nonanalytic in the coupling constant, which is as expected since this solution was not obtained from the perturbative expansion.

5.2 The Schwarzschild Solution in the DGP Model

In the previous section, we examined the vDVZ discontinuity of massive gravity in the context of the spherically symmetric solution for a point mass. We explicitly witnessed the breakdown of the perturbative solution in the limit of vanishing graviton mass. To arrive at a solution in the small graviton mass regime which smoothly transitions to the weak-field Schwarzschild solution of 4D massless gravity, we examined the field equations in the vanishing mass limit and kept up to second-order field contributions. In this regime, we found such a solution which differs from that of the perturbative solution. In this section, we present a similar method to the generally covariant, 5D DGP model which also suffers from the vDVZ discontinuity of massive gravity. We choose a spherically symmetric, 5 dimensional metric ansatz with additional off-diagonal metric contributions. By keeping up to second-order field contributions for the off-diagonal field, we arrive at a set of non-linear, coupled field equations which can be solved. We obtain solutions for the fields which interpolate between the near and far regime. In the far regime, the solution corresponds to that of the linearized perturbative solution which has a 5D tensor structure and a 4D distance dependence. In the near regime, the second-order, off-diagonal terms give a non-vanishing contribution and one arrives at a solution which smoothly transitions to the 4D Einstein theory in the decoupling limit.

The DGP model [8] describes a 3-brane on the boundary of a five-dimensional bulk space Σ. The action is

$$S = M^3 \int_\Sigma d^4x dy \sqrt{-G} \mathcal{R}^{(5)} + \overline{M}^2 \int_{\partial\Sigma} d^4x \sqrt{-g}\, \mathcal{R}^{(4)} \tag{5.18}$$

where $\mathcal{R}^{(5)}$ ($\mathcal{R}^{(4)}$) is the five-(four-) dimensional Ricci Scalar. The solution to the linearized equations bares a striking resemblence to the vDVZ solution of massive gravity and shares the apparent discontinuity in the decoupling limit ($M \to 0$). Porrati [36] argued that this solution is only valid in a limited domain, as was the vDVZ solution of massive gravity, and breaks down in the regime

$$r \lesssim r_c \quad , \quad r_c = \left(\frac{m\overline{M}^2}{18\pi M^6} \right)^{1/3} \tag{5.19}$$

when a static spherically-symmetric source of mass m is considered.

We seek to obtain a solution to the field equations for a static point source with stress-energy tensor

$$T_{AB} = m\delta^0_A \delta^0_B \delta^3(\vec{x})\delta(y) \tag{5.20}$$

which is valid throughout the region $r_m \lesssim r \lesssim 1/m_b$, where

$$r_m = \frac{2m}{8\pi\overline{M}^2} \tag{5.21}$$

is the Schwarzschild radius and

$$m_b = \frac{M^3}{\overline{M}^2} \tag{5.22}$$

is a crossover scale between four-dimensional and five-dimensional behavior.

We choose the *ansatz* for the metric

$$ds^2 = -e^{2B(r,y)}dt^2 + e^{2C(r,y)}\delta_{ij}dx^i dx^j + 2A_i(r,y)dx^i dy + e^{2D(r,y)}dy^2 \tag{5.23}$$

To arrive at a set of compatible field equations, we shall keep first-order contributions in the diagonal components B, C, D, and up to *second-order* terms in the off-diagonal field $\vec{A}$. It is also convenient to introduce the notation

$$\vec{A} = \vec{\nabla}\phi \ , \ \ \Psi' = \frac{1}{r}(\phi')^2 \tag{5.24}$$

where prime denotes differentiation with respect to r, the distance from the point source. A dot will be used for bulk derivatives (with respect to y).

First, let us discuss the lowest order contributions to the field equations in the bulk. The yy component reads

$$2\partial_i\partial^i(B + 2C) + \partial_j(\partial^j\phi\partial_i\partial^i\phi) - \partial_j(\partial^i\phi\partial_i\partial^j\phi) = 0 \tag{5.25}$$

The mixed components are

$$\partial_i(\dot{B} + 2\dot{C}) - \partial^j\phi\partial_j\partial_i\dot{\phi} = 0 \tag{5.26}$$

The spatial brane worldvolume components are

$$(\partial_i\partial_j - \delta_{ij}\partial_k\partial^k)(B + C + D - \dot{\phi}) - \delta_{ij}(\ddot{B} + 2\ddot{C})$$

$$+\partial_k(\partial^k\phi\partial_i\partial_j\phi) - \partial_j(\partial^k\phi\partial_i\partial_k\phi) - \tfrac{1}{2}\delta_{ij}\left(\partial_k(\partial^k\phi\partial_l\partial^l\phi) - \partial_k(\partial^l\phi\partial_l\partial^k\phi)\right) = 0 \tag{5.27}$$

and finally, the tt component is

$$\partial_i\partial^i(2C + D - \dot{\phi}) + 3\ddot{C} + \tfrac{1}{2}\left[\partial_k(\partial^k\phi\partial_i\partial^i\phi) - \partial_j(\partial^k\phi\partial_k\partial^j\phi)\right] = 0 \tag{5.28}$$

In terms of the field Ψ (eq. (5.24)), the field equation (5.25) becomes linear,

$$B + 2C + \Psi = 0 \tag{5.29}$$

Then the mixed components (5.26) may be written as

$$\partial_i \dot{\Psi} + \partial^j \phi \partial_j \partial_i \dot{\phi} = 0 \tag{5.30}$$

whose general solution is

$$\phi'(r,y) = \frac{\alpha(y)}{r^2} + \beta(r) \tag{5.31}$$

We shall make use of gauge freedom to choose $\alpha(y) = 0$, i.e., we demand ϕ (as well as Ψ) be independent of y,

$$\dot{\phi} = \dot{\Psi} = 0 \tag{5.32}$$

This is true to lowest order; higher-order corrections will introduce a non-vanishing $\dot{\phi}$ (and $\dot{\Psi}$).

The remaining field equations (5.27) and (5.28) also become linear. They read, respectively,

$$B + C + D + \tfrac{1}{2}\Psi = 0 \tag{5.33}$$

$$\nabla^2(2C + D + \Psi) + 3\ddot{C} = 0 \tag{5.34}$$

where we used (5.32).

Eqs. (5.29), (5.32) and (5.33) yield

$$C = -\tfrac{1}{2}(B + \Psi) \quad , \quad D = -\tfrac{1}{2}B \tag{5.35}$$

Then eq. (5.34) becomes

$$\ddot{B} + \nabla^2 B = 0 \tag{5.36}$$

whose solution is easily obtained after Fourier-transforming the worldvolume coordinates

$$\tilde{B}(p,y) = \tilde{B}(p,0)e^{-py} \tag{5.37}$$

The bulk behavior of the other fields is then found from (5.35),

$$\tilde{C}(p,y) = -\tfrac{1}{2}\tilde{B}(p,0)e^{-py} - \tfrac{1}{2}\tilde{\Psi}(p) \quad , \quad \tilde{D}(p,y) = -\tfrac{1}{2}\tilde{B}(p,0)e^{-py} \tag{5.38}$$

Having obtained the functional dependence of all fields on y in terms of data on the brane ($y = 0$), we now turn to solving the boundary field equations.

On the boundary ($y = 0$), the spatial brane components yield

$$2M^3\phi + \overline{M}^2(B + C) = 0 \tag{5.39}$$

whereas the tt component is

$$6M^3\dot{C} + 2\overline{M}^2\partial_i\partial^i C + 2M^3\partial_i\partial^i\phi = -m\delta^3(\vec{x}) \tag{5.40}$$

The first term in eq. (5.40) may be dropped in the regime of interest, $p \gg m_b = M^3/\overline{M}^2$. Eliminating C by using (5.35), we obtain

$$\overline{M}^2\nabla^2(B + \Psi) - 2M^3\nabla^2\phi = m\delta^3(\vec{x}) \tag{5.41}$$

Solving for B on the boundary, we find

$$B(r,0) = -\Psi + 2m_b\phi - \frac{m}{4\pi\overline{M}^2 r} \tag{5.42}$$

Using this and (5.35) to eliminate B and C from the other boundary eq. (5.39), we deduce

$$3m_b\phi - \Psi - \frac{m}{8\pi\overline{M}^2 r} = 0 \tag{5.43}$$

Differentiating with respect to r and using (5.24),

$$3m_b\phi' - \frac{1}{r}(\phi')^2 + \frac{m}{8\pi \overline{M}^2 r^2} = 0 \tag{5.44}$$

which can easily be solved for ϕ' yielding,

$$\phi' = \frac{3}{2}m_b r\left(1 - \sqrt{1 + \frac{r_c^3}{r^3}}\right) \tag{5.45}$$

where r_c is given by eq. (5.19). We also have

$$\Psi' = \frac{1}{r}(\phi')^2 = \frac{m}{8\pi \overline{M}^2 r^2} + \frac{9}{4}m_b^2 r\left(1 - \sqrt{1 + \frac{r_c^3}{r^3}}\right) \tag{5.46}$$

Differentiating eq.(5.42) with respect to r, we obtain the form of the field B on the brane,

$$B'(r,0) = \frac{m}{8\pi \overline{M}^2 r^2} - \frac{3}{4}m_b^2 r\left(1 - \sqrt{1 + \frac{r_c^3}{r^3}}\right) \tag{5.47}$$

Summarizing, eqs. (5.35), (5.45) and (5.47) provide the form of the metric on the brane. This solution is valid everywhere on the brane (in the regime $r_m \lesssim r \lesssim 1/m_b$). The solution in the bulk is given by eqs. (5.37) and (5.38) where the Fourier transform $\tilde{B}(p,0)$ is deduced from (5.47).

Next, we examine the near and far regimes, seperated by the crossover distance r_c (eq. (5.19)).

In the far regime ($r \gtrsim r_c$), we have from (5.45),

$$\phi = \frac{m}{24\pi M^3 r}\left\{1 + \mathcal{O}(r_c^3/r^3)\right\} \tag{5.48}$$

and so (using eq. (5.24) or (5.46)),

$$\Psi = -\frac{m}{128\pi \overline{M}^2 r} \left\{ \frac{r_c^3}{r^3} + \mathcal{O}(r_c^6/r^6) \right\} \tag{5.49}$$

Notice that Ψ is of higher order. Therefore, eq. (5.35) implies to lowest order

$$C(r,y) = D(r,y) = -\frac{1}{2}B(r,y) \tag{5.50}$$

Using (5.47), we obtain

$$B(r,0) = -\frac{m}{6\pi \overline{M}^2 r} \left\{1 + \mathcal{O}(r_c^3/r^3)\right\} \tag{5.51}$$

and after Fourier transforming,

$$\tilde{B}(p,0) = -\frac{2m}{3\overline{M}^2 p^2} \left\{1 + \mathcal{O}(p^3/p_c^3)\right\} \tag{5.52}$$

where $p_c \sim 1/r_c$. The y dependence of the fields to lowest order is given by

$$\tilde{C}(p,y) = \tilde{D}(p,y) = -\frac{1}{2}\tilde{B}(p,y) = \frac{m}{3\overline{M}^2 p^2} e^{-py} \tag{5.53}$$

Now taking the inverse Fourier transforms, we finally obtain for $r \gtrsim r_c$,

$$\begin{aligned}
B_>(r,y) &= -\frac{m}{6\pi \overline{M}^2 r} \left\{1 - \frac{2}{\pi}\tan^{-1}\frac{y}{r}\right\} \left(1 + \mathcal{O}(r_c^3/r^3)\right) \\
C_>(r,y) &= \frac{m}{12\pi \overline{M}^2 r} \left\{1 - \frac{2}{\pi}\tan^{-1}\frac{y}{r}\right\} \left(1 + \mathcal{O}(r_c^3/r^3)\right) \\
D_>(r,y) &= \frac{m}{12\pi \overline{M}^2 r} \left\{1 - \frac{2}{\pi}\tan^{-1}\frac{y}{r}\right\} \left(1 + \mathcal{O}(r_c^3/r^3)\right) \\
\phi_>(r,y) &= \frac{m}{24\pi M^3 r} \left(1 + \mathcal{O}(r_c^3/r^3)\right)
\end{aligned} \tag{5.54}$$

This solution corresponds to that of the standard perturbative expansion. Notice that in the decoupling limit ($M^3 \to 0$), ϕ diverges. But this is a limit beyond our approximation above, because corrections become infinite ($r_c \to \infty$, from eq. (5.19)). We next obtain the solution in the near regime which corresponds to the four-dimensional Schwarzschild solution and is valid in the decoupling limit.

In the near regime ($r \lesssim r_c$), we obtain from eqs. (5.45) and (5.46), respectively,

$$\phi = -\sqrt{\frac{mr}{2\pi \overline{M}^2}}\ \left\{1 + \mathcal{O}((r/r_c)^{3/2})\right\} \quad , \quad \Psi = -\frac{m}{8\pi \overline{M}^2 r}\ \left\{1 + \mathcal{O}((r/r_c)^{3/2})\right\} \tag{5.55}$$

In this case Ψ contributes to lowest order. Using (5.47), we deduce on the brane

$$\tilde{B}(p,0) = -\frac{m}{2\overline{M}^2 p^2}\ \left\{1 + \mathcal{O}((p_c/p)^{3/2})\right\} \tag{5.56}$$

The y-dependence to lowest order is given by

$$\begin{aligned}
\tilde{B}(p,y) &= -\frac{m}{2\overline{M}^2 p^2} e^{-py} \\
\tilde{C}(p,y) &= \frac{m}{4\overline{M}^2 p^2}(1 + e^{-py}) \\
\tilde{D}(p,y) &= \frac{m}{4\overline{M}^2 p^2} e^{-py}
\end{aligned} \tag{5.57}$$

where we used (5.37) and (5.38). At $y = 0$, we recover the Schwarzschild solution and therefore agreement with the standard Newtonian potential of massless gravity. All fields are now non-singular in the decoupling limit $M \to 0$. Fourier transforming, in the regime $r_m \lesssim r \lesssim r_c$, we obtain from eqs. (5.55) and (5.57),

$$\begin{aligned}
B_<(r,y) &= -\frac{m}{8\pi \overline{M}^2 r}\ \left\{1 - \frac{2}{\pi}\tan^{-1}\frac{y}{r}\right\}\left(1 + \mathcal{O}((r/r_c)^{3/2})\right) \\
C_<(r,y) &= \frac{m}{8\pi \overline{M}^2 r}\ \left\{1 - \frac{1}{\pi}\tan^{-1}\frac{y}{r}\right\}\left(1 + \mathcal{O}((r/r_c)^{3/2})\right) \\
D_<(r,y) &= \frac{m}{16\pi \overline{M}^2 r}\ \left\{1 - \frac{2}{\pi}\tan^{-1}\frac{y}{r}\right\}\left(1 + \mathcal{O}((r/r_c)^{3/2})\right)
\end{aligned}$$

$$\phi_<(r,y) \quad = \quad -\sqrt{\frac{mr}{2\pi\overline{M}^2}}\ \left(1+\mathcal{O}((r/r_c)^{3/2})\right) \tag{5.58}$$

To summarize the results of this chapter, we examined the spherically symmetric solution of a point mass in both the Pauli-Fierz model of massive gravity and the 5D DGP model, both of which suffer from the vDVZ discontinuity at linear order. Following closely to the work of Vainshtein [32] for the case of PF massive gravity in the vanishing graviton mass regime, we kept all lowest-order field contributions which includes second-order terms in one of the fields. In this small graviton mass regime, we obtained a solution which corresponds to that of the 4D Einstein solution. For the 5D DGP model, we derived a perturbative expansion which, as in the PF massive case, also yields a solution which reduces to that of the Schwarzschild solution in a decoupling limit. By keeping second-order terms of the off-diagonal metric components of our metric ansatz, we arrived at an explicit solution both on the brane and in the bulk. On the brane, our solution interpolates between the near and far regimes which are separated by the distance scale r_c (eq. (5.19)); the critical radius r_c found in [36] is determined using this formalism. At distances below the critical radius r_c, the perturbative expansion yields the four-dimensional Schwarzschild solution, demonstrating the absence of the van Dam-Veltman-Zakharov (vDVZ) discontinuity [30, 31]. In the far regime at distances above the critical radius r_c, our solution reduces to that found in the linear perturbative expansion.

Chapter 6

Conclusion

In Chapter 2, we introduced a generic model of Brane Induced Gravity for a 3-brane residing in an infinite-volume Minkowski bulk-space. We limited our analysis to the case of a delta-function type brane where the stress-energy tensor was chosen to reside on the brane and has only 4D worldvolume components. By fine-tuning the 4D cosmological constant to exactly cancel the brane tension, we arrived at a Minkowski brane background. In a simplest setup scenario, the model is that of a bulk-space action of a D-dimensional Ricci scalar which generates D-dimensional Einstein equations. Due to the interactions of the bulk gravitons with the stress-energy tensor confined to the brane, we added an induced 4D Ricci tensor to the bulk action and arrived at the model proposed by Dvali-Gabadadze-Porrati, commonly referred to as the DGP Model in the literature. By varying the action and perturbatively expanding around the Minkowski background, we arrived at the linearized DGP field equations. Choosing a D-dimensional harmonic gauge, we solved the field equations and obtained the solution for the graviton and scalar propagator for the cases of $D = 5$ and $D > 5$. For $D = 5$ dimensions, the solution gives rise to a 4D $1/r$ potential plus a logarithmic repulsive term in the near regime which corresponds to tensor-scalar gravity. In the far regime, we obtained a 5D Newtonian-like potential. For the case of $D > 5$ dimensions, we showed that the solution for graviton propagator on the brane has a tensor structure and distance dependence of exactly 4D Einstein gravity. The bulk-space exhibits the characteristic of infrared transparency where only the $p^2 = 0$ mode gives rise to non-zero interactions between matter placed in the bulk and matter localized on the brane.

In Chapter 3, we generalized the model by allowing the 3-brane to have a finite thickness extending into the bulk-space. This finite thickness can arise if the brane is treated as a smooth soliton in the bulk or by transverse fluctuations of the brane into the bulk-space which gives rise to an effective brane thickness. In the previous chapter when we examined a delta-function type brane, we arrived at the solution for the graviton propagator which has the exact tensor structure and distance dependence of the 4D Einstein solution, however, this solution was obtained in a singular manner. Giving the brane a finite thickness into the bulk regulates the model and allows for a careful examination of the solution. Following closely to the delta-function case, we expanded the DGP field equations around a Minkowski background and obtained the solution for the graviton propagator. We first examined the pole structure and found that the graviton propagator contains an infinite towers of massive gravitons and tachyonic ghosts. In the limit of the fat brane becoming thin, the two terms which gave rise to the massive and tachyonic poles become vanishingly small and the solution for the graviton propagator reduces to that of 4D Einstein gravity, as was found for the treatment of a delta-function type brane. We then analyzed the tensor structure of the momentum dependent graviton propagator for this brane of finite thickness. In the small momentum regime, the graviton propagator exhibited a D-dimensional behavior, which was in contrast to the large momentum regime (above the critical scale p_c (eq. (3.51)) but well below the inverse brane width Λ), where the contributions from the massive gravitons and tachyonic ghosts conspired to produce a propagator on the brane whose tensor structure and distance dependence was that of four-dimensional Einstein gravity.

In Chapter 4, we examined a general theory of linearized 4D massive gravity which allowed for all possible combinations of the metric perturbations, parameterized by free parameters, which gives rise to local massive graviton contributions to the field equations. After varying the action and obtaining the field equations, we expanded around a Minkowski background and found the solution for the metric perturbations. By examing the pole structure of the metric perturbations, we found that the free parameters were severely constrained when one insists on a well-defined theory which is free of tachyonic and ghost-like states; this brought us to the 4D Pauli-Fierz model of massive gravity. We showed that the solution for the metric perturbations of the PF model suffers from a van Dam-Veltman-Zakharov (vDVZ) discontinuity where one does not arrive at the solution for 4D massless Einstein gravity in the limit of vanishing graviton mass. We then examined the 5D DGP model which also suffers from a vDVZ discontinuity at linear order, which is due to the breakdown of the weak field itself. By including two additional linear action contributions to the DGP model parameterized by bulk and brane free parameters, we arrived at a generalized, regulated DGP model which cures the vDVZ discontinuity by changing the linearized DGP field equations. We solved the coupled field equations and arrived at a solution for the metric perturbations which were written in terms of the brane parameters. We showed that the solution exhibits the expected crossover behavior and is independent of the free parameters; in the near regime the metric perturbations have the exact tensor structure and distance dependence of the 4D theory whereas in the far regime the solution is that of a 5D theory. We rewrote the solution for the metric perturbations revealing the pole structure and examined the parameter space. We found that the region of the parameter space which yielded non-physical resonances corresponding to intermediate, metastable states and is free of tachyonic-type resonances.

In Chapter 5, we readdressed the vDVZ discontinuity of the Pauli-Fierz model and the 5D DGP model by examining the spherically symmetric solution of a massive point source in the respective setups. By choosing a spherically symmetric metric ansatz, we obtained the full, non-linear field equations. Expanding to linear order, we explicitly witnessed the breakdown of the perturbative expansion which does not allow for a smooth transition to the 4D solution in the vanishing graviton mass limit. In this chapter, we wished to obtain the solution in the small graviton mass regime which does smoothly transition to the 4D solution and do so by keeping up to higher-order field contributions. For the case of PF massive gravity in the vanishing graviton mass regime, we kept all lowest-order field contributions which includes second-order terms in one of the fields. Following [32], we obtained a solution in this small graviton mass regime which corresponds to that of the 4D Einstein solution and showed that the vDVZ discontinuity can thus be avoided by the inclusion of higher-order terms. For the 5D DGP model, we adopted a spherically symmetric ansatz with the addition of an off-diagonal metric contribution. We expanded the DGP field equations keeping all first-order field contributions and up to second-order terms in the off-diagonal field. We obtained the solution which is valid throughout the desired regime and examined the solution in both distance regimes. In the near regime, this solution yielded a 4D distance dependence and metric tensor and was found to reduce to the 4D Schwarzschild solution in the decoupling limit. In the far regime, the solution was found to have a 5D tensor structure and a 4D distance dependence. In this regime, the solution corresponds to that of the linear perturbative expansion.

The work that has been presented here amounts to a small part of the current research on braneworld scenarios. After the discovery of the importance of D-branes in '95 by Polchinski [58], there has emerged a wealth of intensive research on Brane Induced Gravity (BIG) and the Randall -Sundrum (RS) [6, 7] scenarios of warped extra dimensions, both offering an attractive alternative to compactification. Brane Induced Gravity is attractive in that it successfully offers an explanation for the weakness of gravity. Open strings, which represent spin-1 and -1/2 standard model particles, are confined to the brane through Dirichlet boundary conditions whereas closed strings, representing spin-2 gravitons, have no such imposed constraints. Spin-2 gravitons reside in both the bulk and on the brane and appear weak to a worldvolume brane observer due to this spreading out.

Brane Induced Gravity also offers an alternative explanation to the recent data which suggests that our universe is expanding at an accelerated rate. Instead of hypothesizing dark energy to successfully account for the repulsive force driving the acceleration of expansion of the universe, the large extra dimensional scenarios modify the gravitational theories at large-distances with the gravitational effects of the extra dimensions emerging on the distance scale of the cosmological horizons. As a result, the Newtonian force law becomes inherently higher dimensional in the large distance regime, thus, gravity gets weaker at cosmological distances. The cosmological solution found in [11] describes a universe accelerated beyond the crossover scale. This acceleration takes place despite the fact that there is no cosmological constant. Bulk gravity sees its own induced curvature term on the brane as a cosmological constant and accelerates [12]. We would like to extend our work further by exploring some issues in cosmology.

We conclude this thesis by commenting that braneworld scenarios residing in large extra dimensions are still in their infancy with much work remaining. As was found in [22, 29] and discussed in Chapter 3, the flat space propagator exhibits tachyonic poles with negative residues. The position of these poles and their existence is UV regularization dependent. Currently, it is not clear whether these poles would remain in a consistent UV completed theory [26]. It would be interesting to see if these negative norm states of the fat brane scenario persist in a de Sitter background and whether this could correspond to a true vacuum of the quantum theory.

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Appendix

Appendix

The solution for the metric perturbations for 3-brane of finite thickness embedded in a D-dimensional bulk (5.18) is found in terms of a scalar propagator which satisfies the inhomogenous wave equation

$$\left[M^{D-2}\left(p^2-\Box_N\right)-(\lambda-1)\,\overline{M}^2 p^2\sigma_\Lambda(y)\right]\mathcal{G}_\lambda(p,y)=\sigma_\Lambda(y) \tag{A1}$$

The solution to this equation is found by first solving the related Green function equation

$$\left[M^{D-2}\left(p^2-\Box_N\right)-(\lambda-1)\,\overline{M}^2 p^2\sigma_\Lambda(y)\right]G_\lambda(p,\vec{y},\vec{y}\,')=\delta^{D-4}(y-y') \tag{A2}$$

where

$$\mathcal{G}_\lambda(p,y)=\int d^{D-4}y'\;\sigma_\Lambda(y')\;G_\lambda(p,\vec{y},\vec{y}\,')\;\;,\;\;\lambda=\frac{3(D-4)}{(D-2)} \tag{A3}$$

Eq.(A2) can be rewritten into the form

$$\left[\frac{1}{y^{D-5}}\frac{d}{dy}\left(y^{D-5}\frac{d}{dy}\right)-\frac{\Lambda^2}{y^2}+k_\lambda^2 p^2\right]G_\lambda(p,\vec{y},\vec{y}\,')=-\frac{1}{M^{D-2}}\delta^{D-4}(y-y') \tag{A4}$$

where we used explicitly written the bulk laplacian in hyperspherical coordinates with Λ^2 the bulk angular momentum operater. In addition, we have defined k_λ^2 in terms of step-function form of the density function $\sigma_\lambda(y)$ (3.7)

$$\begin{aligned} k_\lambda^2 \ |_{y<1/\Lambda} &= (\lambda-1)\frac{\overline{M}^2\Lambda^{D-4}}{\omega_{D-4}M^{D-2}} - 1 \simeq (\lambda-1)\frac{\overline{M}^2\Lambda^{D-4}}{\omega_{D-4}M^{D-2}} \\ k_\lambda^2 \ |_{y>1/\Lambda} &= -1 \end{aligned} \tag{A5}$$

for k_λ^2 both on the brane and in the bulk. To solve eq.(A4), we can expand the Green function in terms of the hyperspherical harmonics [49]

$$G_\lambda(p,\vec{y},\vec{y}\,') = \sum_{l,m} \mathcal{G}_\lambda(p,y,y\,')Y_{lm}^*(\Omega')Y_{lm}(\Omega) \tag{A6}$$

where $\mathcal{G}_\lambda(p,y,y\,')$ is the radial Green function. The hyperspherical harmonics $Y_{lm}(\Omega)$ are eigenfunctions of the bulk angular momentum operator Λ^2 obeying the eigenvalue equation

$$\left[\Lambda^2 - l(l+d-6)\right] Y_{lm}(\Omega) = 0 \tag{A7}$$

Plugging the expansion for the Green function (A6) into eq.(A4) and using the orthogonality conditions of hyperspherical harmonics given by

$$\int d\Omega \; Y_{l'm'}^*(\Omega) \; Y_{lm}(\Omega) = \delta_{l'l}\delta_{m'm} \tag{A8}$$

we obtain an equation for the radial Green function

$$\left[\frac{1}{y^{D-5}}\frac{d}{dy}\left(y^{D-5}\frac{d}{dy}\right) - \frac{l(l+(D-6))}{y^2} + k_\lambda^2 p^2\right]\mathcal{G}_\lambda(p,y,y\,') = -\frac{1}{M^{D-2}}\frac{\delta(y-y')}{y^{D-5}} \tag{A9}$$

At this point we're left with solving this one-dimensional radial Green function equation. We can however simplify the equation further. We are interested in obtaining the solution to the non-homogenous wave equation obtained by integrating the Green function over the density function (A3). Because of the hyperspherical symmetry of our chosen matter source dictated by the spread function $\sigma_\Lambda(y)$, $\mathcal{G}_\lambda(p,y)$ is easily shown to be dependent only on the radial Green function with $l = m = 0$. Again, plugging the expansion for the Green function (A6) into (A3) and using orthogonality, we obtain

$$\mathcal{G}_\lambda(p,y) = \int d^{D-4}y' \; \sigma_\Lambda(y') G_\lambda(p,\vec{y},\vec{y}\,') = \int dy' \; y'^{D-5} \sigma_\Lambda(y') \mathcal{G}_\lambda(p,y,y') \tag{A10}$$

where the radial Green function obeys the $l = 0$ equation

$$\left[\frac{1}{y^{D-5}}\frac{d}{dy}\left(y^{D-5}\frac{d}{dy}\right) + k_\lambda^2 p^2\right] \mathcal{G}_\lambda(p,y,y\,') = -\frac{1}{M^{D-2}}\frac{\delta(y-y')}{y^{D-5}} \tag{A11}$$

The radial Green function can be found explicitly in terms of Bessel functions. Solving for (A11) on the brane and in the bulk and using the appropriate boundary conditions, the solution is

$$\begin{aligned} \mathcal{G}_\lambda(p,y,y\,')\,|_{y\le 1/\Lambda} &= \frac{i\pi}{4M^{D-2}}\left(\frac{1}{yy\,'}\right)^{(D-6)/2} \frac{1}{\mathcal{A}_\lambda} J_{(D-6)/2}(k_\lambda p y_<) \\ &\times \left[\mathcal{B}_\lambda^{(2)} H^{(1)}_{(D-6)/2}(k_\lambda p y_>) - \mathcal{B}_\lambda^{(1)} H^{(2)}_{(D-6)/2}(k_\lambda p y_>)\right] \end{aligned} \tag{A12}$$

inside the brane ($y \le 1/\Lambda$), and outside the brane,

$$\mathcal{G}_\lambda(p,y,y')\,|_{y>1/\Lambda} = -\frac{1}{M^{D-2}}\left(\frac{1}{yy'}\right)^{(D-6)/2} \frac{\Lambda}{p\mathcal{A}_\lambda}\; J_{(D-6)/2}(k_\lambda p y') K_{(D-6)/2}(py) \tag{A13}$$

where $y_<(y_>)$ is the smaller (larger) of y and y' and $H_N^{(1,2)}(x)$ are Hankel functions

$$\begin{aligned} \mathcal{A}_\lambda &= k_\lambda K_{(D-6)/2}(p/\Lambda) J_{(D-4)/2}(k_\lambda p/\Lambda) - K_{(D-4)/2}(p/\Lambda) J_{(D-6)/2}(k_\lambda p/\Lambda) \\ \mathcal{B}_\lambda^{(1,2)} &= k_\lambda K_{(D-6)/2}(p/\Lambda) H^{(1,2)}_{(D-4)/2}(k_\lambda p/\Lambda) - K_{(D-4)/2}(p/\Lambda) H^{(1,2)}_{(D-6)/2}(k_\lambda p/\Lambda) \end{aligned} \tag{A14}$$

Notice that when $\lambda = 0$, k_λ^2 becomes negative. The Green function equation becomes

$$\left[\frac{1}{y^{D-5}}\frac{d}{dy}\left(y^{D-5}\frac{d}{dy}\right) - \kappa^2 p^2\right]\mathcal{G}_0(p,y,y\,') = -\frac{1}{M^{D-2}}\frac{\delta(y-y')}{y^{D-5}} \qquad \text{(A15)}$$

where

$$\begin{aligned} \kappa^2 \;|_{y\leq 1/\Lambda} = -k_0^2 \;|_{y\leq 1/\Lambda} &= 1 + \frac{\overline{M}^2\Lambda^{D-4}}{\omega_{D-4}M^{D-2}} \\ \kappa^2 \;|_{y>1/\Lambda} = -k_0^2 \;|_{y>1/\Lambda} &= 1 \end{aligned} \qquad \text{(A16)}$$

The radial Green function is in terms of Modified Bessel functions. As before we again solve for (A11) on the brane and in the bulk and use the appropriate boundary conditions. The solution is

$$\begin{aligned} \mathcal{G}_0(p,y,y\,')\;|_{y\leq 1/\Lambda} = & \quad \frac{1}{M^{D-2}}\left(\frac{1}{yy'}\right)^{(D-6)/2}\frac{1}{\mathcal{A}_0}I_{(D-6)/2}(\kappa p y_<) \\ & \times \quad \left[\mathcal{A}_0 K_{(D-6)/2}(\kappa p y_>) + \mathcal{B}_0 I_{(D-6)/2}(\kappa p y_>)\right] \end{aligned} \qquad \text{(A17)}$$

inside the brane ($y \leq 1/\Lambda$), and outside the brane,

$$\mathcal{G}_0(p,y,y\,')\;|_{y>1/\Lambda} = \frac{1}{M^{D-2}}\left(\frac{1}{yy'}\right)^{(D-6)/2}\frac{\Lambda}{p\mathcal{A}_0}I_{(D-6)/2}(\kappa p y')K_{(D-6)/2}(py) \qquad \text{(A18)}$$

where

$$\begin{aligned} \mathcal{A}_0 &= \kappa I_{(D-4)/2}(\kappa p/\Lambda)K_{(D-6)/2}(p/\Lambda) + I_{(D-6)/2}(\kappa p/\Lambda)K_{(D-4)/2}(p/\Lambda) \\ \mathcal{B}_0 &= \kappa K_{(D-4)/2}(\kappa p/\Lambda)K_{(D-6)/2}(p/\Lambda) - K_{(D-6)/2}(\kappa p/\Lambda)K_{(D-4)/2}(p/\Lambda) \end{aligned} \qquad \text{(A19)}$$

where $I_N(x)$ and $K_N(x)$ are the Modified Bessel functions.

We can now easily obtain the values for $\mathcal{G}_\lambda(p,y),\ \mathcal{G}_0(p,y)$ which are needed to acquire the solutions for the metric and scalar perturbations $\tilde{h}_{\mu\nu},\ \tilde{h}^\alpha_\alpha$. Plugging (A12 ,A17) into eq. (A3), a short calculation yields the solutions for the non-homogeneous wave equation.

The solutions to the non-homogeneous wave equation are

$$\begin{aligned}
\mathcal{G}_\lambda(p,y)\,|_{y\le 1/\Lambda} &= -\frac{1}{(\lambda-1)\overline{M}^2 p^2}\left[1+\frac{1}{\mathcal{A}_\lambda}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} K_{(D-4)/2}(p/\Lambda)J_{(D-6)/2}(k_\lambda p y)\right] \\
\mathcal{G}_\lambda(p,y)\,|_{y>1/\Lambda} &= -\frac{1}{(\lambda-1)\overline{M}^2 p^2}\cdot\frac{k_\lambda}{\mathcal{A}_\lambda}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} J_{(D-4)/2}(k_\lambda p/\Lambda)K_{(D-6)/2}(py) \quad \text{(A20)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_0(p,y)\,|_{y\le 1/\Lambda} &= \frac{1}{\overline{M}^2 p^2}\left[1-\frac{1}{\mathcal{A}_0}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} K_{(D-4)/2}(p/\Lambda)I_{(D-6)/2}(\kappa p y)\right] \\
\mathcal{G}_0(p,y)\,|_{y>1/\Lambda} &= \frac{1}{\overline{M}^2 p^2}\cdot\frac{\kappa}{\mathcal{A}_0}\left(\frac{1}{y\Lambda}\right)^{(D-6)/2} I_{(D-4)/2}(\kappa p/\Lambda)K_{(D-6)/2}(py) \qquad \text{(A21)}
\end{aligned}$$

Vita

Chad A. Middleton was born in Kankakee, IL to Don and Lana on May 12, 1975. After graduating from Bradley-Bourbonnais Community High School in 1993, he pursued a Bachelor's degree in Physics from Eastern Illinois University in Charleston, IL. He joined the University of Tennessee in the fall of 1999 to pursue the Doctor of Philosophy degree in Physics and graduated in December 2005. He has accepted the position of Assistant Professor at Rhodes College in Memphis, TN and will begin Fall 2005.