Lecture Series on...

General Relativity and Differential Geometry

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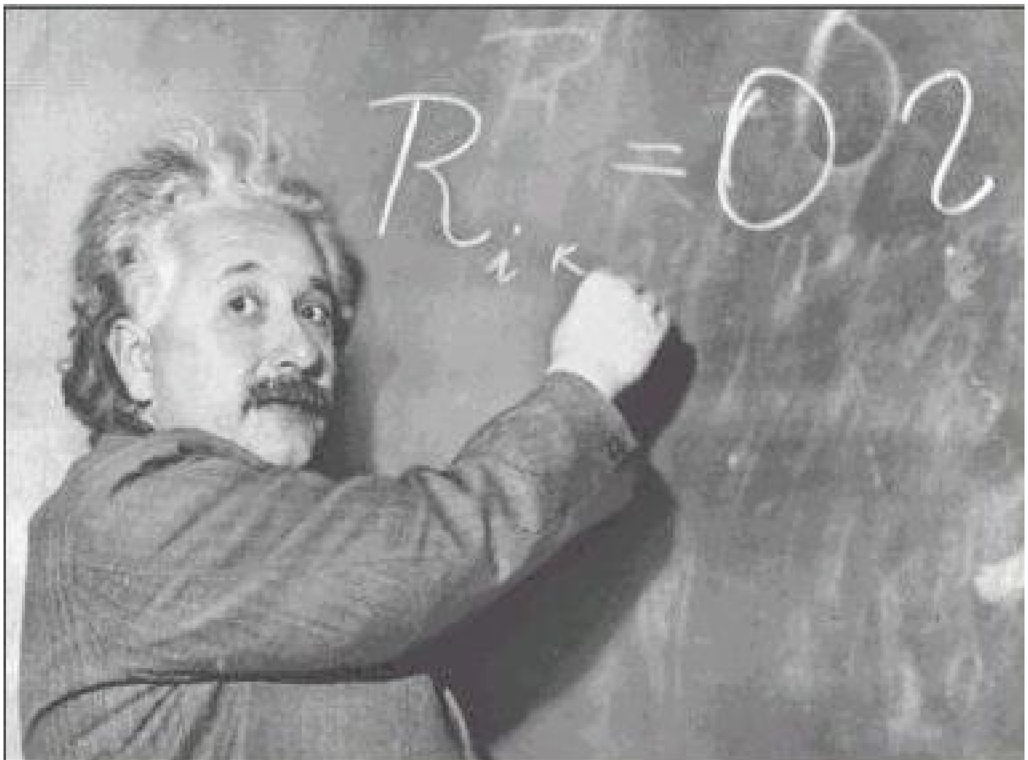
OUTLINE

- Distance in 3D Euclidean Space
- Distance in 4D Minkowski Spacetime
- Principle of Equivalence
- Distance in 4D Non-Euclidean Spacetime (metric tensor, Christoffel symbols, Einstein Field Equations)

The General Theory of Relativity



Einstein and the Ricci Tensor

- "General Relativity is the most beautiful physical theory ever invented."

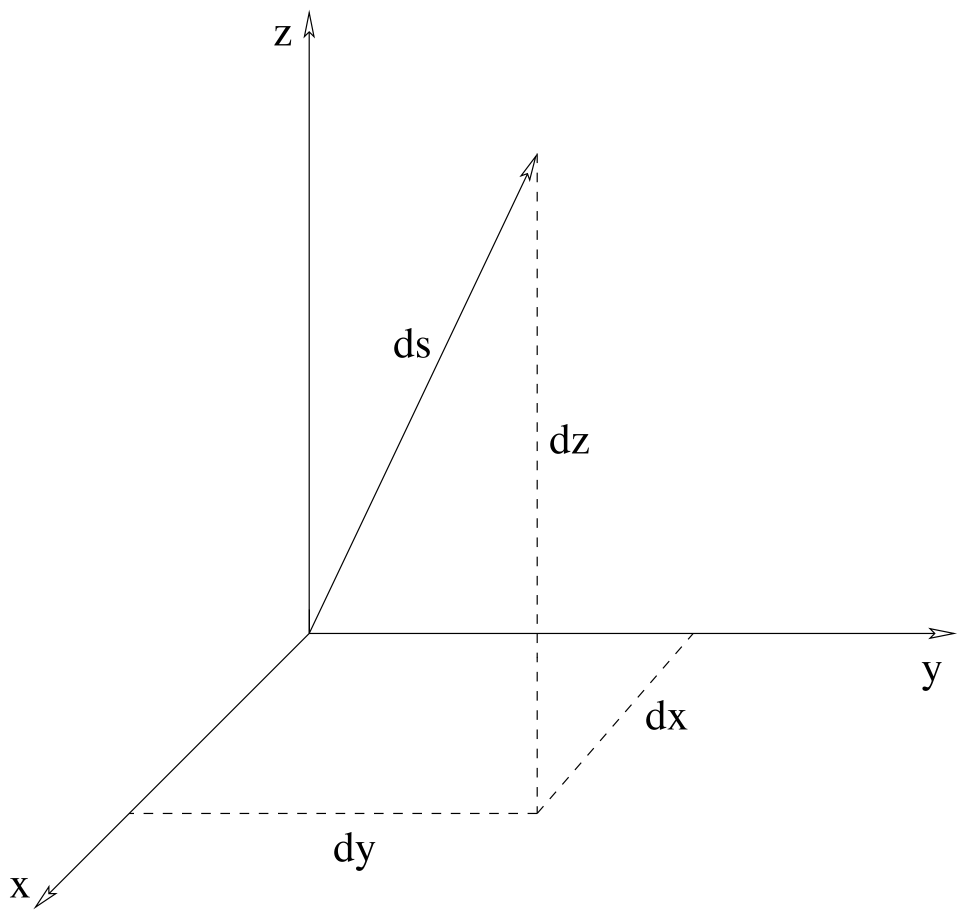
  – *Spacetime and Geometry*

- "One of the Greatest Achievements of the Human Mind."

  – *Introducing Einstein's Relativity*

Line element (distance) in Euclidean space

$$\lim_{\Delta\to 0}(\Delta s)^2 = (\Delta x)^2+(\Delta y)^2+(\Delta z)^2$$
$$ds^2_{3D} = dx^2+dy^2+dz^2 = d\vec{x}\cdot d\vec{x} \tag{1}$$



- ds^2_{3D} is the line-element measuring distance
  $\Rightarrow$ Pythagorean Theorem
- ds^2_{3D} is invariant under rotations

Line element using matrix notation

$$ds^2_{3D} = \delta_{ij} dx^i dx^j$$

where

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

- Matrix multiplication

- Repeated index $\equiv$ sum over index

$$ds^2_{3D} = \delta_{ij} dx^i dx^j = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} dx^i dx^j$$

$$x^{i,j} : \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$ds^2_{3D} = \delta_{ij} dx^i dx^j$$

Summing over the i index...

$$ds^2_{3D} = \delta_{1j} dx^1 dx^j + \delta_{2j} dx^2 dx^j + \delta_{3j} dx^3 dx^j \tag{3}$$

Summing over j index...

$$\begin{aligned} ds^2_{3D} &= \delta_{11} dx^1 dx^1 + \delta_{12} dx^1 dx^2 + \delta_{13} dx^1 dx^3 \\ &= \delta_{21} dx^2 dx^1 + \delta_{22} dx^2 dx^2 + \delta_{23} dx^2 dx^3 \\ &= \delta_{31} dx^3 dx^1 + \delta_{32} dx^3 dx^2 + \delta_{33} dx^3 dx^3 \end{aligned} \tag{4}$$

$$\begin{aligned} \therefore ds^2_{3D} &= dx_1 dx_1 + dx_2 dx_2 + dx_3 dx_3 \\ &= dx^2 + dy^2 + dz^2 \end{aligned} \tag{5}$$

Example:

What is the Euclidean line element in spherical-polar coordinates?

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{6}$$

- Taking the differentials...

$$\begin{aligned} dx &= d(r\sin\theta\cos\phi) \\ &= \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi \end{aligned} \tag{7}$$

- and plugging into the cartesian line element

$$ds^2_{3D} = dx^2 + dy^2 + dz^2$$

- yields the spherical-polar line element

$$ds^2_{3D} = dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2)$$

- Example of a *coordinate transformation*

Example:

What is the line element for the unit 2-sphere?

- locus of points in $\mathbb{R}^3$ at unit distance from origin

- Set $r = 1$ and $dr = 0$

$$ds_2^2 = d\theta^2 + \sin^2\theta d\phi^2$$

- Non-Euclidean manifold

Special Theory of Relativity *Postulates:*

- All inertial observers are equivalent
- $c = constant$

Consequences:

- Time and length are *relative* quantities
  - ds^2_{3D} is NOT invariant

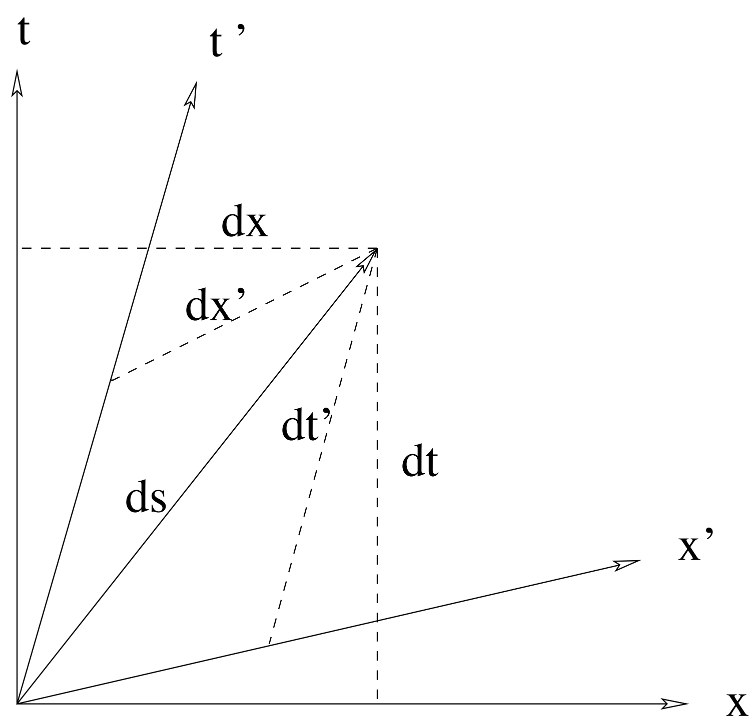
$$ds'^2_{3D} \neq ds^2_{3D}$$

<u>Lorentz transformations</u>

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx/c^2) \end{aligned} \tag{8}$$

Line element in Minkowski space

$$
\begin{aligned}
ds^2 &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \\
&= -c^2dt^2 + d\vec{x} \cdot d\vec{x} \qquad (9)
\end{aligned}
$$



- ds^2 is the line element measuring length
- ds^2 is invariant under rotations

$$
ds^2 = ds'^2 \qquad (10)
$$

Proper time:

$$ds^2 = ds'^2 \tag{11}$$

- For proper time, set $d\vec{x}' = 0$

$$-c^2 d\tau^2 = -c^2 dt^2 + d\vec{x} \cdot d\vec{x}$$

solving for $d\tau$

$$\begin{aligned} d\tau &= dt\sqrt{1 - v^2/c^2} \quad \text{where} \quad v^2 = \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt} \\ &= \frac{dt}{\gamma} \end{aligned} \tag{12}$$

Line element using matrix notation

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

- Again, repeated index $\equiv$ sum over index

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$$

$$x^{\mu,\nu} : \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

- $\eta_{\mu\nu}$ is the Minkowski metric

- Newton's Second Law

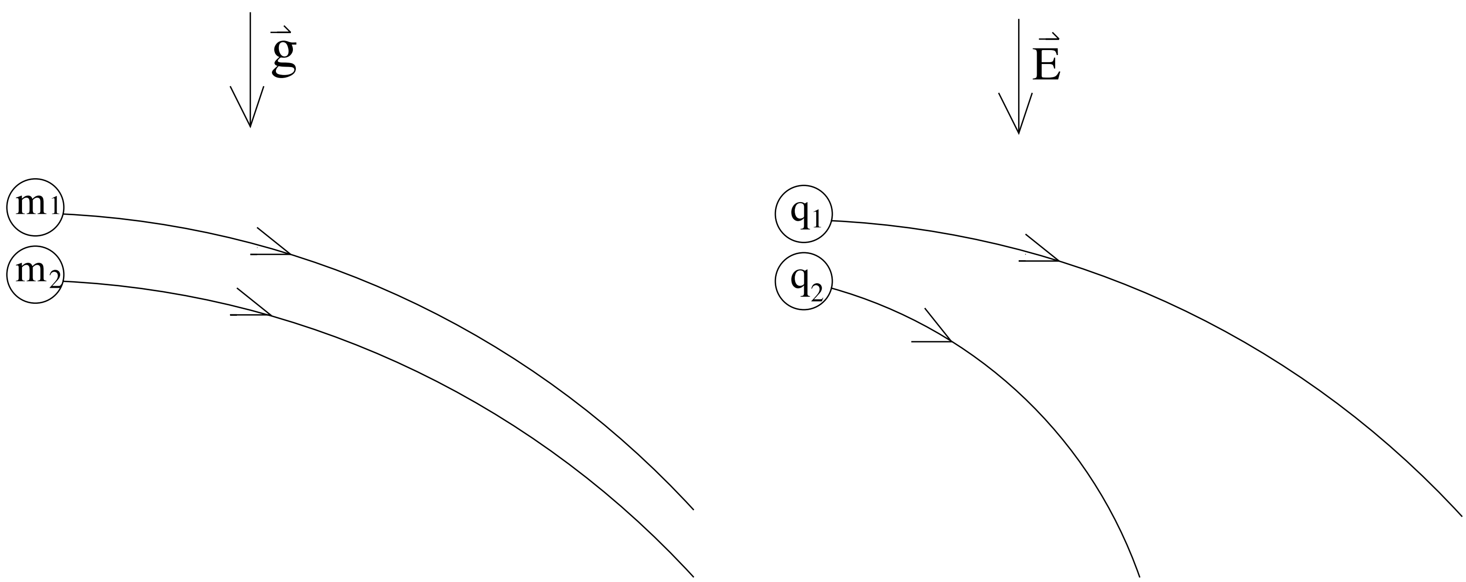
$$\vec{F} = m_i \vec{a}$$

- Newton's Law of Gravitation (uniform field)

$$\vec{F}_g = -m_g \nabla \phi \, (= m_g \vec{g})$$

- Electric Field

$$\vec{F}_e = Q\vec{E}$$



$\vec{a} = (m_g/m_i)\vec{g}$ $\vec{a} = (Q/m_i)\vec{E}$

- *Eötvös experiment* verified that

$$m_g = m_i$$

to 1 part in 10^{12}!

The *Weak Equivalence Principle* is

$$\vec{g} = \vec{a}$$

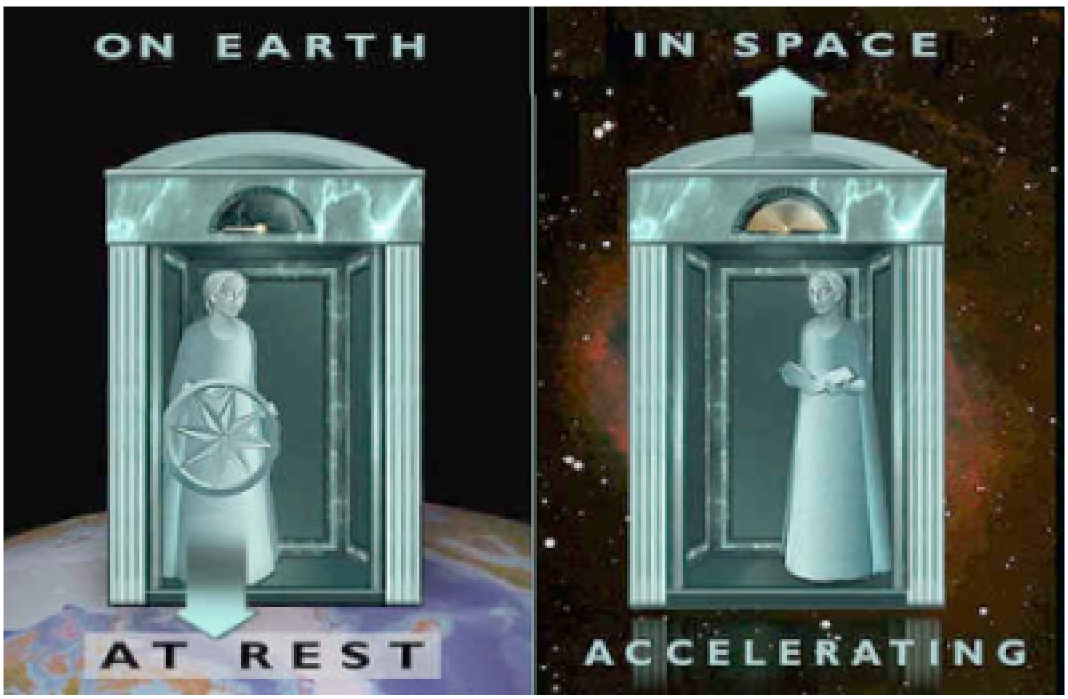
- All objects "fall" at the same rate

(independent of the mass).

- Suggests a preferred class of trajectories through spacetime.

$\rightarrow$*inertial* (freely-falling) trajectories

The motion of freely-falling particles is the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime.



– What about *massless* particles?

- Elevator observer sees light moving in a *curved* path

– Is Earth observer inertial?

- Elevator observer is *non-inertial*

  ∴ Earth observer is *non-inertial*!

Coincidence(?) in Newtonian theory

- All *inertial forces* have the mass as a

  constant of proportionality in them.

  ...as does the *gravitational force*.

- Einstein suggested that we treat gravitation as an inertial effect (from not using an inertial frame).

In Minkowski coordinates in Special Relativity, the equation for a test particle is

$$\frac{d^2x^\mu}{d\tau^2} = 0$$

For a non-inertial frame of reference, the equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

the additional terms are the *inertial force* terms.

- By principle of equivalence, the gravitational forces should be given by an appropriate $\Gamma^\mu_{\alpha\beta}$.

- *Geodesic equation* - Free particles move along paths of "shortest possible distances"

Line element using *curved* space

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- $g_{\mu\nu}$ is the metric tensor
- $\eta_{\mu\nu}$ is the *flat* limit of $g_{\mu\nu}$
- $g_{\mu\nu} = g_{\nu\mu}$

- Again, repeated index $\equiv$ sum over index

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$

$$\begin{aligned} ds^2 \ = \ & g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + g_{02} dx^0 dx^2 + ... \\ & + ... + g_{13} dx^1 dx^3 + ... + g_{33} dx^3 dx^3 \end{aligned} \tag{14}$$

- $g_{\mu\nu}$ defines the geometry of spacetime
- *Know* $g_{\mu\nu}$*, Know Geometry*

Example:

What is the metric tensor of the unit 2-sphere?

$$ds_2^2 = d\theta^2 + \sin^2\theta d\phi^2$$

The metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad (15)$$

or writing the components explicitly...

$$g_{\theta\theta} = 1\ ,\ g_{\theta\phi} = g_{\phi\theta} = 0\ ,\ g_{\phi\phi} = \sin^2\theta$$

What is the inverse metric? $(g \cdot g^{-1} = 1)$

$$(g_{\mu\nu})^{-1} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}$$

with components ...

$$g^{\theta\theta} = 1\ ,\ g^{\theta\phi} = g^{\phi\theta} = 0\ ,\ g^{\phi\phi} = 1/\sin^2\theta$$

- All the ways curvature manifests itself rely on something called a "connection".

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma} \right)$$

where

$$\partial_\beta \equiv \frac{\partial}{\partial x^\beta}$$

- δ is a repeated index $\rightarrow$ Sum!

- Looks like a tensor... but it's not a tensor.

Notice:

$$\Gamma = \Gamma(g, \partial g)$$

- Christoffel is a function of the metric and the derivative of the metric

Example:

What are the possible connections for the unit 2-sphere?

$$ds_2^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Components of the metric:

$$g_{\theta\theta} = 1\ ,\ g_{\theta\phi} = g_{\phi\theta} = 0\ ,\ g_{\phi\phi} = \sin^2\theta$$

Answer:

$$\Gamma^{\theta}_{\theta\theta}\ ,\ \Gamma^{\theta}_{\theta\phi}\ ,\ \Gamma^{\phi}_{\theta\theta}\ ,\ \Gamma^{\theta}_{\phi\phi}\ ,\ \Gamma^{\phi}_{\theta\phi}\ ,\ \Gamma^{\phi}_{\phi\phi}$$

What are the connections for the unit 2-sphere?

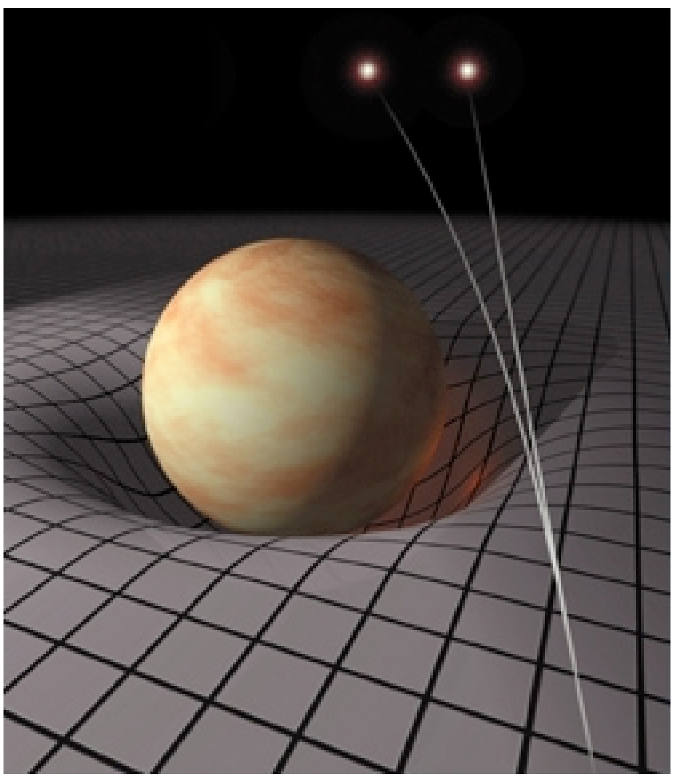
$$\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}g^{\theta\delta}\left(\partial_\phi g_{\phi\delta} + \partial_\phi g_{\phi\delta} - \partial_\delta g_{\phi\phi}\right) \\ &= \frac{1}{2}g^{\theta\theta}\left(2\partial_\phi g_{\phi\theta} - \partial_\theta g_{\phi\phi}\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial\theta}(\sin^2\theta) = -\sin\theta\cos\theta \end{aligned} \tag{16}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta\ ,\ \Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\phi} = \Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi} = 0$$

General Theory of Relativity

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor describing the *curvature* of space.
- $T_{\mu\nu}$ is the stress-energy tensor which describes *matter*.



- *MATTER tells space how to CURVE and SPACE tells matter how to MOVE!*

Einstein Field Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

- Subscripts label elements of each matrix
- Set of 10 second-order, non-linear, partial differential equations
- EM, Strong, & Weak $\rightarrow$ fields on spacetime

  $\Rightarrow$ Gravity is curvature of spacetime itself!

Einstein Field Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

where

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}$$

is the Ricci tensor and

$$R = g^{\mu\nu}R_{\mu\nu}$$

is the Ricci scalar.

Notice:

$$R_{\mu\nu} = R_{\mu\nu}(\Gamma, \partial\Gamma) \ \text{ but } \ \Gamma = \Gamma(g, \partial g)$$

The Einstein Tensor

$$G_{\mu\nu} = G_{\mu\nu}(g, \partial g, \partial^2 g)$$

is written entirely in terms of the metric tensor!

Example: Line element for 2-sphere of *radius* a

$$ds_2^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^{\phi}_{\theta\phi} &= \cot\theta \\
0 &= \Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\phi} = \Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi}
\end{aligned} \tag{17}$$

What are the Ricci tensors?

$$R_{\phi\phi} = \partial_\alpha \Gamma^{\alpha}_{\phi\phi} - \partial_\phi \Gamma^{\alpha}_{\phi\alpha} + \Gamma^{\alpha}_{\phi\phi}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\phi\alpha}\Gamma^{\alpha}_{\beta\phi}$$

- Summing out indicies...

$$\begin{aligned}
R_{\phi\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\beta}_{\theta\beta} - \Gamma^{\beta}_{\phi\theta}\Gamma^{\theta}_{\beta\phi} - \Gamma^{\beta}_{\phi\phi}\Gamma^{\phi}_{\beta\phi} \\
&= \partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} \\
&= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}
\end{aligned} \tag{18}$$

- Plugging in the functions

$$\begin{aligned} R_{\phi\phi} &= \partial_\theta(-\sin\theta\cos\theta) + \cot\theta \cdot \sin\theta\cos\theta \\ &= \sin^2\theta \end{aligned} \tag{19}$$

- Likewise,

$R_{\theta\theta} = 1$

What is the Ricci scalar?

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= \frac{1}{a^2\sin^2\theta} \cdot \sin^2\theta + \frac{1}{a^2} \\ &= \frac{2}{a^2} \end{aligned} \tag{20}$$

The Ricci scalar...

- ... completely characterizes curvature (2D).
- ... is constant on 2-sphere.
- ... decreases for increasing a.

In 1916, Karl Schwarzschild presented an EXACT solution!

- Choose a point mass

$$T_{\mu\nu} = m\delta^0_\mu \delta^0_\nu \delta^3(\vec{x})$$

- Choose metric ansatz

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}d\vec{x} \cdot d\vec{x}$$

- Plug metric ansatz into the Einstein Field equation

The solution is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the differential solid angle.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2$$

- When $r \rightarrow r_{sch} = 2GM \;\Rightarrow\; dt^2 \rightarrow 0$ & $dr^2 \rightarrow \infty$
  - Schwarzschild radius
    $\Rightarrow$ Not a real singularity
- When $r \rightarrow 0 \;\Rightarrow\; dt^2 \rightarrow \infty$ & $dr^2 \rightarrow 0$
  - Spacetime singularity!
- Solution predicts Black holes!