Elliptical-like orbits on a warped spandex fabric

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Kepler’s 3 Laws of planetary motion

Kepler’s 1\textsuperscript{st} Law..
• The planets move in \textit{elliptical} orbits with the sun at one focus.

Kepler’s 2\textsuperscript{nd} Law..
• A line extending from the Sun to any planet sweeps out \textit{equal areas in equal times}.

Kepler’s 3\textsuperscript{rd} Law..

\[ T^2 = \left( \frac{4\pi^2}{G} \right) \cdot \frac{r^3}{M} \]
Einstein’s theory of general relativity

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

- \( G_{\mu\nu} \) describes the curvature of spacetime
- \( T_{\mu\nu} \) describes the matter & energy in spacetime

Matter tells space how to curve, space tells matter how to move.
Is there a 2D surface that will generate orbits that obey Kepler’s 3 laws?

For a marble orbiting on a warped elastic fabric, how does the *period* of the orbit relate to the *radial distance*?

Can one generate *elliptical-like orbits* on the warped elastic surface?


The physics of the analogy...

Is there a 2D surface that will generate orbits that obey Kepler’s 3 laws?

-> Not exactly!*

For a marble orbiting on a warped elastic fabric, how does the period of the orbit relate to the radial distance?

-> See*

Can one generate elliptical-like orbits on the warped elastic surface?

-> Yes, stay tuned!**


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For a marble orbiting on a warped elastic fabric, how does the period of the orbit relate to the radial distance?

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Can one generate elliptical-like orbits on the warped elastic surface?

-> Yes, stay tuned!

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Outline

• A marble rolling on a cylindrically symmetric surface (Lagrangian dynamics)
• The shape of the spandex fabric (Calculus of Variations)
• Small slope regime
  o Angular separation between successive apsides
  o Experiment
• Large slope regime
  o Angular separation between successive apsides
  o Experiment
• Elliptical-like orbits in GR
A marble rolling on a cylindrically symmetric surface

• is described by a Lagrangian of the form..

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) + \frac{1}{2} \omega^2 - mgz \]

• Now, for the rolling marble..

\[ I = \frac{2}{5} mR^2 \quad \text{and} \quad \omega^2 = v^2 / R^2 \quad \text{so} \quad \frac{1}{2} I \omega^2 = \frac{1}{5} mv^2 \]

Notice:

• The marble is constrained to reside on the fabric..

\[ z = z(r) \]
The Lagrange equations of motion

• take the form...
  
  \[ (1 + z'^2) \ddot{r} + z' z'' \dot{r}^2 - r \dot{\phi}^2 + \frac{5}{7} g z' = 0 \]

• define the differential operator...
  
  \[ \frac{d}{dt} = \frac{5\ell}{7r^2} \frac{d}{d\phi} \]

• * becomes...
  
  \[ (1 + z'^2) \frac{d^2 r}{d\phi^2} + \left( z' z'' - \frac{2}{r} (1 + z'^2) \right) \left( \frac{dr}{d\phi} \right)^2 - r + \frac{7g}{5\ell^2} \cdot z' r^4 = 0 \]
The equation of motion for a rolling marble on a cylindrically symmetric surface...

\[
(1 + z'^2) \frac{d^2 r}{d\phi^2} + (z' z'' - \frac{2}{r} (1 + z'^2)) \left( \frac{dr}{d\phi} \right)^2 - r + \frac{7g}{5\ell^2} \cdot z' r^4 = 0
\]

For elliptical-like orbits with small eccentricities...

\[ r(\phi) = r_0 (1 - \varepsilon \cos(\nu \phi)) \]

- where \( \nu \) is the precession parameter
  \[ \nu \equiv \frac{360^\circ}{\Delta \phi} \]
- Inserting the approximate solution into the equation of motion yields
  \[ \nu = \sqrt{\frac{3z_0' + z_0''r_0}{z_0' + z_0'^3}} \]
The slope of the spandex fabric

Technique:

   
i. **Elastic PE of the spandex.**
   
ii. **Gravitational PE of the spandex.**
   
iii. **Gravitational PE of the central mass.**

2. Apply **Calculus of Variations.**

   ⇒ The elastic fabric-mass system will assume the shape which *minimizes* the *total* PE of the system.

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*i.* and *iii.* first considered in “Comment on “The shape of ‘the Spandex’ and orbits upon its surface”, Am. J. Phys. 70 (10), 1056-1058 (2002)
The slope of the spandex fabric

The Euler-Lagrange equation can be integrated once and takes the form:

\[ r z' \left[ 1 - \frac{1}{\sqrt{1 + z'^2}} \right] = \alpha(M + \sigma_0 \pi r^2) \]

where we defined the parameter \( \alpha \equiv \frac{g}{2\pi E} \).
The slope of the spandex fabric

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\[ \alpha \equiv \frac{g}{2\pi E} \]

- mass of central object
- areal mass density
- Modulus of elasticity
The angular separation & the slope of the spandex fabric

\[ \Delta \phi = 360^\circ \sqrt{\frac{z'_0 (1 + z''_0^2)}{3z'_0 + r_0 z''_0}} \]

\[ rz' \left[ 1 - \frac{1}{\sqrt{1 + z'f^2}} \right] = \alpha (M + \sigma_0 \pi r^2) \]

- Angular separation between successive like-apsides
- The slope of the spandex fabric

- Small slope regime..

\[ z'(r) \ll 1 \quad \text{so} \quad \frac{1}{\sqrt{1 + z'^2}} = 1 - \frac{1}{2} z'^2 + o(z'^4) \]

- Large slope regime..

\[ z'(r) \gg 1 \quad \text{so} \quad \frac{1}{\sqrt{1 + z'^2}} = \frac{1}{z'} \frac{1}{\sqrt{1 + 1/z'^2}} = \frac{1}{z'} \left( 1 - \frac{1}{2z'^2} + o(1/z'^4) \right) \]
Elliptical-like orbits on the spandex fabric

- 4 ft. diameter trampoline frame
  - styrofoam insert for zero pre-stretch
  - truck tie down around perimeter
- Camera mounted directly above, ramp mounted on frame.
- Position determined every 1/60 s and average radius, $r_{\text{ave}}$, calculated per $\Delta \phi$.
- $\Delta \phi$ from $r_{\text{max}}$ to $r_{\text{max}}$ can be measured.
- $\Delta \phi$ can be calculated via...

\[
\Delta \phi = 360^\circ \sqrt{\frac{z_0'(1 + z_0'^2)}{3z_0' + r_0z_0''}}
\]

The three orbits imaged here were found to have angular separations between successive apocenters of $\Delta \phi = 213.5^\circ, 225.7^\circ, 223.9^\circ$ and eccentricities of $\varepsilon = 0.31, 0.29, 0.31$.
Elliptical-like orbits in the small slope regime

• For \( z'(r) \ll 1 \), the slope of the spandex surface takes the form...

\[ z'(r) \approx \left(\frac{2\alpha}{r}\right)^{1/3} (M + \sigma_0 \pi r^2)^{1/3} \]

• Plugging this into the equation determining the angular separation yields a theoretical value of...

\[ \Delta \phi = 220^\circ \left[ 1 + \left(\frac{2\alpha (M + \sigma_0 \pi r^2)}{r_0}\right)^{2/3} \right]^{1/2} \left[ 1 + \frac{1}{4} \frac{\sigma_0 \pi r^2}{(M + \sigma_0 \pi r^2)} \right]^{-1/2} \]
Elliptical-like orbits in the large slope regime

• For $z'(r) \gg 1$, the slope of the spandex surface takes the form...

\[ z'(r) \approx 1 + \frac{\alpha M}{r} \]

• Plugging this into the equation determining the *angular separation* yields a theoretical value of...

\[ \Delta \phi = 360^\circ \sqrt{\frac{(1 + \alpha M/r_0)}{(3 + 2\alpha M/r_0)} \left[ 1 + \left( 1 + \frac{\alpha M}{r_0} \right)^2 \right]} \]
Elliptical-like orbits in the large slope regime

Notice:

- For a large central mass and a very small average radial distance, the angular separation yields a limiting behavior

\[
\lim_{\alpha M/r_0 \gg 1} \Delta \phi \simeq \frac{360^\circ}{\sqrt{2}} \cdot \frac{\alpha M}{r_0}
\]

- when \( \alpha M/r_0 > \sqrt{2} \), \( \Delta \phi > 360^\circ \)!
Elliptical-like orbits with small eccentricities in GR

The equation of motion for an object of mass \( m \) orbiting about a spherically symmetric object of mass \( M \), in the presence of a cosmological constant (or vacuum energy), \( \Lambda \), is

\[
\ddot{r} + \frac{GM}{r^2} - \frac{\ell^2}{r^3} + \frac{3GM\ell^2}{c^2r^4} - \frac{1}{3} \Lambda c^2 r = 0
\]

- define the differential operator..

\[
\frac{d}{d\tau} = \frac{\ell}{r^2} \frac{d}{d\phi}
\]

- * becomes

\[
\frac{d^2r}{d\phi^2} - \frac{2}{r} \left( \frac{dr}{d\phi} \right)^2 + \frac{GM}{\ell^2} \frac{r^2}{r^2} - r + \frac{3GM}{c^2} - \frac{\Lambda c^2}{3\ell^2} r^5 = 0
\]
Elliptical-like orbits with small eccentricities in GR

The orbital equation of motion...

\[ \frac{d^2 r}{d\phi^2} - \frac{2}{r} \left( \frac{dr}{d\phi} \right)^2 + \frac{GM}{\ell^2} r^2 - r + \frac{3GM}{c^2} - \frac{\Lambda c^2}{3\ell^2} r^5 = 0 \]

For elliptical-like orbits with small eccentricities...

\[ r(\phi) = r_0(1 - \varepsilon \cos(\nu \phi)) \]
Precessing elliptical orbits in GR with small eccentricities

Case I: $\Lambda = 0$

• We find the solution, to 1st order in the eccentricity, when..

\[
\ell^2 = GMr_0 \left(1 - \frac{3GM}{c^2r_0}\right)^{-1}
\]
\[
\nu^2 = 1 - \frac{6GM}{c^2r_0}
\]

Notice:
Innermost stable circular orbit

• When $r_0 < 6GM/c^2 \equiv r_{ISCO}$, $\nu$ becomes complex: *elliptical-like* orbits not allowed!
• When $r_0 < 3GM/c^2$, $\nu$ & $\ell$ become complex: *no circular orbits.*
Case I: $\Lambda = 0$

- The angular separation between successive apocenters takes the form...

\[ \Delta \phi = 360^\circ \left(1 - \frac{r_{ISCO}}{r_0}\right)^{-1/2} \quad \text{where} \quad r_{ISCO} \equiv \frac{6GM}{c^2} \]

- expand about...

\[ r_0 \equiv r_{ISCO} + r \quad \text{where} \quad r \ll r_{ISCO} \]

- * becomes...

\[ \lim_{r \ll \frac{6GM}{c^2}} \Delta \phi \simeq 360^\circ \sqrt{6} \cdot \sqrt{\frac{GM/c^2}{r}} \]
**Precessing elliptical orbits in GR with small eccentricities**

Case I: $\Lambda = 0$, The behavior of the angular separation...

- in GR, near the *innermost stable circular orbit*...

\[
\lim_{r \ll 6GM/c^2} \Delta \phi \simeq 360^\circ \sqrt{6} \cdot \sqrt{\frac{GM/c^2}{r}}
\]

- of the marble, in the large slope regime...

\[
\lim_{r_0 \ll \alpha M} \Delta \phi \simeq \frac{360^\circ}{\sqrt{2}} \cdot \frac{\alpha M}{r_0}
\]

Notice:

- Both expressions *diverge* in the limit of vanishingly small distances.
- $\alpha$ plays the role of $G/c^2$; both set the scale of their respective theories.
Precessing elliptical orbits in GR with small eccentricities

Case II: $\Lambda \neq 0$

• We find the solution, to 1st order in the eccentricity, when...

\[
\begin{align*}
\ell^2 &= Gr_0 \left(1 - \frac{3GM}{c^2r_0}\right)^{-1} (M - 2 \cdot \frac{4}{3} \pi r_0^3 \cdot \rho_0) \\
\nu^2 &= 1 - \frac{6GM}{c^2r_0} - 6 \left(1 - \frac{3GM}{c^2r_0}\right) \frac{4}{3} \pi r_0^3 \cdot \rho_0 \\
&\quad (M - 2 \cdot \frac{4}{3} \pi r_0^3 \cdot \rho_0)
\end{align*}
\]

• The angular separation between successive apocenters takes the form...

\[
\Delta \phi = 360^\circ \left(1 - \frac{6GM}{c^2r_0}\right)^{-1/2} \left[1 - 6 \left(\frac{1 - 3GM/c^2r_0}{1 - 6GM/c^2r_0}\right) \frac{4}{3} \pi r_0^3 \cdot \rho_0 \\
&\quad (M - 2 \cdot \frac{4}{3} \pi r_0^3 \cdot \rho_0)\right]^{-1/2}
\]
Case II: $\Lambda \neq 0$, The behavior of the angular separation...

- in GR, about a static, spherically symmetric massive object in the presence of a constant vacuum energy.

$$
\Delta \phi = 360^\circ \left(1 - \frac{6GM}{c^2 r_0}\right)^{-1/2} \left[1 - 6 \left(\frac{1 - 3GM/c^2 r_0}{1 - 6GM/c^2 r_0}\right) \frac{\frac{4}{3} \pi r_0^3 \cdot \rho_0}{(M - 2 \cdot \frac{4}{3} \pi r_0^3 \cdot \rho_0)}\right]^{-1/2}
$$

- of the marble, in the small slope regime...

$$
\Delta \phi = 220^\circ \left[1 + \left(2 \alpha (M + \sigma_0 \pi r^2)/r_0\right)^{2/3}\right]^{1/2} \left[1 + \frac{1}{4} \frac{\sigma_0 \pi r^2}{(M + \sigma_0 \pi r^2)}\right]^{-1/2}
$$

Notice:
- The areal mass density, $\sigma_o$, of the spandex fabric plays the role of a negative vacuum energy density, $-\rho_o$, of spacetime.
Conclusion

- We find good agreement between theory and experiment for the angular separation between successive apsides, $\Delta \phi$.

- In the large slope regime, $\Delta \phi > 360^\circ$ for small radii and large central mass.

- $\Delta \phi$ diverges in the limit of vanishing small distances for
  - the marble in the large slope regime.
  - a particle near the innermost stable circular orbit.

- The areal mass density, $\sigma_o$, of the spandex fabric plays the role of a negative vacuum energy density, $-\rho_o$, of spacetime.
Einstein’s theory of general relativity

Consider a static, spherically symmetric, massive object...

Embedding diagram \((t = t_0, \theta = \pi/2)\).

- 2D equatorial ‘slice’ of the 3D space at one moment in time

\[
z(r) = 2\sqrt{\frac{2GM}{c^2}} \left( r - \frac{2GM}{c^2} \right)
\]

Does a warped spandex fabric yield orbits that obey Kepler’s 3 laws?

where \(2GM/c^2 = 1\)
Circular orbits on a warped spandex fabric, revisited...

Small slope regime:

\[ T^3 = \left( \frac{28\pi^2}{5g} \right)^{3/2} \frac{1}{\sqrt{2\alpha}} \cdot \frac{r^2}{(M + \pi\sigma_0 r^2)^{1/2}} \]

Large slope regime:

\[ T = \left( \frac{28\pi^2}{5g} \right)^{1/2} \cdot \frac{r}{(M\alpha + r')^{1/2}} \]

Elastic PE of the spandex fabric

Elastic PE of a differential concentric ring of the fabric of unstretched width $dr$...

$$dU_e = \frac{1}{2} \kappa (\sqrt{dr^2 + dz^2} - dr)^2$$

- Define the modulus of elasticity, $E$ ..

$$E = \frac{\kappa dr}{2\pi r}$$

- Integrating the differential segment over the whole fabric, the total elastic PE of the fabric is...

$$U_e = \int_0^R \pi E \cdot r (\sqrt{1 + (\frac{dz}{dr})^2} - 1)^2 \, dr$$
Gravitational PE of the spandex fabric

Gravitational PE of a differential concentric ring of the fabric.

\[ dU_{g,s} = dm_s g \cdot z \]

- The mass of the differential ring is a constant under stretching.

\[ dm_s = \sigma_0 \cdot 2\pi r dr = \sigma(z') \cdot 2\pi r \sqrt{dr^2 + dz^2} \]

where \( \sigma_0, \sigma(z') \) are the unstretched, variable areal mass densities.

- Integrating the differential segment over the whole fabric, the total gravitational PE of the fabric is.

\[ U_{g,s} = \int_0^R 2\pi \sigma_0 g \cdot rz \ dr \]
Gravitational PE of the central mass

\[ U_{g,M} = Mg \cdot z(0) = - \int_0^R Mg \cdot z'(r) \, dr \]

Notice:
- we approximate the central mass as being point-like.
The total PE of the spandex-central mass system

\[ U = U_e + U_{g,s} + U_{g,M} = \int_0^R f(z, z'; r) \, dr \]

where we defined the functional..

\[ f(z, z'; r) \equiv \pi E \cdot r (\sqrt{1 + z'^2} - 1)^2 + 2\pi \sigma_0 g \cdot rz - Mg \cdot z' \]

To minimize the total PE, subject to the Euler-Lagrange eqn..

\[ \frac{\partial f}{\partial z} - \frac{d}{dr} \frac{\partial f}{\partial z'} = 0 \]
The shape equation for the elastic fabric

The Euler-Lagrange equation takes the form..

\[
\frac{d}{dr} \left[ rz' \left[ 1 - \frac{1}{\sqrt{1 + z'^2}} \right] - \frac{M g}{2\pi E} \right] = \frac{\sigma_0 g}{E} \cdot r
\]

• which can be integrated..

\[
rz' \left[ 1 - \frac{1}{\sqrt{1 + z'^2}} \right] = \alpha (M + \pi \sigma_0 r^2)
\]

• where we defined the parameter..

\[
\alpha \equiv \frac{g}{2\pi E}
\]