Circular orbits on a warped spandex fabric

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Einstein’s theory of general relativity

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

- \( G_{\mu\nu} \) describes the curvature of spacetime
- \( T_{\mu\nu} \) describes the matter & energy in spacetime

Matter tells space how to curve, space tells matter how to move.
Einstein’s theory of general relativity

Consider a spherically-symmetric, non-rotating massive object...

Embedding diagram \((t = t_0, \theta = \pi/2)\).

- 2D equatorial ‘slice’ of the 3D space at one moment in time

\[
z(r) = 2\sqrt{2M (r - 2M)}
\]

Is there a warped 2D surface that will yield the orbits of planetary motion?

Kepler’s 3\textsuperscript{rd} Law for planetary motion

- Newton’s 2\textsuperscript{nd} Law..
  \[
  \frac{GmM}{r^2} = \frac{mv^2}{r}
  \]
- using the relation..
  \[
  v = \frac{2\pi r}{T}
  \]
- yields Kepler’s 3\textsuperscript{rd} Law..
  \[
  T^2 = \left( \frac{4\pi^2}{G} \right) \cdot \frac{r^3}{M}
  \]

Notice:
- Kepler’s 3\textsuperscript{rd} Law is \textit{independent} of \(m\)!
Outline

• A marble rolling on a cylindrically-symmetric surface (Lagrangian dynamics)
• The shape of the spandex fabric (Calculus of Variations)
• Small curvature regime
  o Kepler-like expression
  o Experimentation
• Large curvature regime
  o Kepler-like expression
  o Direct measurement of the modulus of elasticity
  o Experimentation
• Circular orbits in GR
A marble rolling on a cylindrically-symmetric surface

• is described by a Lagrangian of the form..

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) + \frac{1}{2} I \omega^2 - mgz \]

• Now, for the marble..

\[ I = \frac{2}{5} mR^2 \quad \text{and} \quad \omega^2 = \frac{v^2}{R^2} \quad \text{so} \quad \frac{1}{2} I \omega^2 = \frac{1}{5} mv^2 \]

Notice:
• The marble is constrained to reside on the fabric..

\[ z = z(r) \]
The Lagrange equation of motion

for the radial-coordinate..

\[ \frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = 0 \]

• yields the equation of motion for the marble..

\[(1 + z'^2)\ddot{r} + z' z'' \dot{r}^2 - r \dot{\phi}^2 + \frac{5}{7} g z' = 0 \]*

• compare to the equation of motion for *planetary orbits*..

\[ \ddot{r} - r \dot{\phi}^2 + \frac{GM}{r^2} = 0 \]

* will NOT yield *Newtonian-like orbits of planetary motion* for a marble on ANY cylindrically-symmetric surface!

Circular orbits on a cylindrically-symmetric surface

For the equation of motion for the marble:

\[(1 + z'^2)\ddot{r} + z' z'' \dot{r}^2 - r \dot{\phi}^2 + \frac{5}{7} gz' = 0\]

- setting \(\dot{r} = \ddot{r} = 0\) for circular orbits, we obtain:

\[\frac{4\pi^2 r}{T^2} = \frac{5}{7} g \cdot z'(r)\]

Notice:
- we used the relation \(v = r \dot{\phi} = 2\pi r / T\)
- depends \textit{linearly} on the \textit{slope} of the spandex fabric.
The shape of the spandex fabric

Technique:
   i. Elastic PE of the spandex.
   ii. Gravitational PE of the spandex.
   iii. Gravitational PE of the central mass.

2. Apply Calculus of Variations.
   ⇒ The elastic fabric-mass system will assume the shape which minimizes the total PE of the system.

i. and iii. first considered in “Comment on “The shape of ‘the Spandex’ and orbits upon its surface”, Am. J. Phys. 70 (10), 1056-1058 (2002)
Elastic PE of the spandex fabric

Elastic PE of a differential concentric ring of the fabric of unstretched width $dr$...

$$dU_e = \frac{1}{2} \kappa (\sqrt{dr^2 + dz^2} - dr)^2$$

- Define the modulus of elasticity, $E$..

$$E = \frac{\kappa dr}{2\pi r}$$

- Integrating the differential segment over the whole fabric, the total elastic PE of the fabric is...

$$U_e = \int_{0}^{R} \pi E \cdot r (\sqrt{1 + z'^2} - 1)^2 \, dr$$
Gravitational PE of the spandex fabric

Gravitational PE of a differential concentric ring of the fabric.

\[ dU_{g,s} = dm_s g \cdot z \]

- The mass of the differential ring is a constant under stretching.

\[ dm_s = \sigma_0 \cdot 2\pi r dr = \sigma(z') \cdot 2\pi r \sqrt{dr^2 + dz^2} \]

where \( \sigma_0, \sigma(z') \) are the unstretched, variable areal mass densities.

- Integrating the differential segment over the whole fabric, the total gravitational PE of the fabric is.

\[ U_{g,s} = \int_0^R 2\pi \sigma_0 g \cdot rz \, dr \]
Gravitational PE of the central mass

\[ U_{g,M} = Mg \cdot z(0) = - \int_0^R Mg \cdot z'(r) \, dr \]

Notice:
• we approximate the central mass as being point-like.
The *total* PE of the spandex-central mass system

\[ U = U_e + U_{g,s} + U_{g,M} = \int_0^R f(z, z'; r) \, dr \]

where we defined the functional..

\[ f(z, z'; r) \equiv \pi E \cdot r(\sqrt{1 + z'^2} - 1)^2 + 2\pi \sigma_0 g \cdot rz - M g \cdot z' \]

To minimize the *total* PE, subject to the Euler-Lagrange eqn..

\[ \frac{\partial f}{\partial z} - \frac{d}{dr} \frac{\partial f}{\partial z'} = 0 \]
The shape equation for the elastic fabric

The Euler-Lagrange equation takes the form...

\[ \frac{d}{dr} \left[ rz' \left[ 1 - \frac{1}{\sqrt{1 + z'^2}} \right] - \frac{Mg}{2\pi E} \right] = \frac{\sigma_0 g}{E} \cdot r \]

• which can be integrated..

\[ rz' \left[ 1 - \frac{1}{\sqrt{1 + z'^2}} \right] = \alpha(M + \pi \sigma_0 r^2) \]

• where we defined the parameter..

\[ \alpha \equiv \frac{g}{2\pi E} \]
The circular equation of motion & the shape equation

\[ \frac{4\pi^2 r}{T^2} = \frac{5}{7} g \cdot z'(r) \]

\[ rz' \left[ 1 - \frac{1}{\sqrt{1 + z'^2}} \right] = \alpha \left( M + \pi \sigma_0 r^2 \right) \]

- Circular equation of motion
- The shape equation

• Small curvature regime..
  \[ z'(r) \ll 1 \quad \text{so} \quad \frac{1}{\sqrt{1 + z'^2}} = 1 - \frac{1}{2} z'^2 + o(z'^4) \]

• Large curvature regime..
  \[ z'(r) \gg 1 \quad \text{so} \quad \frac{1}{\sqrt{1 + z'^2}} = \frac{1}{z'} \frac{1}{\sqrt{1 + 1/z'^2}} = \frac{1}{z'} \left( 1 - \frac{1}{2z'^2} + o(1/z'^4) \right) \]
The small curvature regime

When \( z'(r) \ll 1 \) ...

- expanding the shape equation and inserting into the circular eqn of motion

\[
T^3 = \left( \frac{28\pi^2}{5g} \right)^{3/2} \frac{1}{\sqrt{2\alpha}} \cdot \frac{r^2}{(M + \pi\sigma_0 r^2)^{1/2}}
\]

Notice:

- \( T^3 \propto r^2/\sqrt{M} \) * when \( M \gg \pi\sigma_0 r^2 \)
- \( T^3 \propto r/\sqrt{\sigma_0} \) when \( M \ll \pi\sigma_0 r^2 \)
- Two competing terms on equal footing when \( M \simeq \pi\sigma_0 r^2 \approx 0.10 \text{ kg} \)

The experiment in the small curvature regime

- 4 ft. diameter trampoline frame
  - styrofoam insert for zero pre-stretch
  - truck tie down around perimeter
- Camera mounted directly above, ramp mounted on frame
- Most circular video clip (of ~12) imported into Tracker
- Position determined every 1/30 s and average radius, \( r_{ave} \), calculated per revolution
- Shift by 1/8 revolution for subsequent data point
The experiment in the small curvature regime

Plot of $T^3$ vs $r_{ave}^2/(M+\pi\sigma_0 r_{ave}^2)^{1/2}$ ...

Notice:

- orbit for zero central mass!
- slope = 45.6 kg$^{1/2}$ s$^3$/m$^2$ w/ $R^2 = 0.994$

The slope yields a value for $\alpha$ and $E$...

$$\alpha = \frac{1}{2} \left( \frac{28\pi^2}{5g} \right)^3 \cdot \frac{1}{\text{slope}^2} \simeq 0.043 \text{ m/kg}$$

$$E = \frac{g}{2\pi\alpha} \simeq 36 \text{ N/m}$$

Central Mass
- 0 kg
- 0.067 kg
- 0.199 kg
- 0.535 kg
- 0.601 kg
The large curvature regime

When \( z'(r) \gg 1 \) ...

- expanding the shape equation and inserting into the circular eqn of motion

\[
T = \left( \frac{28\pi^2}{5g} \right)^{1/2} \cdot \frac{r}{(M\alpha + r)^{1/2}}
\]

Notice:

- \( T \propto r/\sqrt{M} \) * when \( r \ll M\alpha \)
- When \( r \gg M\alpha \), \( z'(r) \approx 1 \), so above equation invalid!
- Using \( \alpha = 0.043 \text{ m/kg} \), we get poor results!

* corresponds to the solution of the 2D Laplace equation with cylindrical-symmetry
Direct measurement of the modulus of elasticity, $E$

The shape equation in the large curvature regime is..

$$z'(r) \approx \frac{M \alpha}{r} + 1$$

- integrating yields..

$$\frac{z(M)}{\ln(R_B)} = (M - M_0)\alpha$$

Notice:
- Plot of $z(M)/\ln(R_B)$ vs $(M - M_0)$ yields the slope, which is the value of $\alpha$!

Top 10 diamonds:
$M = 0.274\text{kg} - 1.174\text{kg}$ in $0.1\text{ kg}$ intervals

Bottom 14 diamonds:
$M = 1.274\text{kg} - 7.774\text{kg}$ in $0.5\text{ kg}$ intervals
Direct measurement of the modulus of elasticity, $E$

Plot of $z(M)/\ln(R_B)$ vs $(M - M_0)$ yields the slope, which is the value of $\alpha$!

Notice:
- $M = 0.274 - 0.674 \text{ kg regime}$  
  $\alpha \approx 0.030 \text{ m/kg}$
- $M = 5.274 - 7.774 \text{ kg regime}$  
  $\alpha \approx 0.006 \text{ m/kg}$
The experiment in the large curvature regime

Plot of $T$ vs $r_{ave}/(M\alpha + r_{ave})^{1/2}$.

- $\text{slope}_{exp} = 2.62 \text{ s/m}^{1/2}$
- $\text{slope}_{th} = \left(\frac{28\pi^2}{5g}\right)^{1/2} = 2.37 \text{ s/m}^{1/2}$

Notice:
- $\sim 10\%$ error for $\alpha = 0.006 \text{ kg/m}$, which compares with $\sim 94\%$ error when $\alpha = 0.043 \text{ kg/m}$. 
Circular orbits in GR

The metric exterior to a spherically-symmetric object of mass $M$, in the presence of a cosmological constant (or vacuum energy), $\Lambda$.

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2$$

where

$$\rho_{vac} = \frac{\Lambda}{8\pi G}$$

Notice..

- $\Lambda > 0$, Schwarzschild – de Sitter spacetime
- $\Lambda < 0$, Schwarzschild – Anti-de Sitter spacetime
- $\Lambda = 0$, Schwarzschild solution
Circular orbits in GR

By normalizing the four-velocity and employing conservation of energy and angular momentum..

• The radial equation of motion..

$$\mathcal{E} = \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V_{eff}(r) \text{ where } V_{eff}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2)$$

For circular orbits, set..

• $$\frac{d}{dr}V_{eff}(r) = 0$$

• $$\mathcal{E} = V_{eff}(r)$$
Circular orbits in GR

One arrives at an exact Kepler-like expression of the form..

\[ T^2 \propto \frac{r^3}{(M - \frac{\Lambda}{3G} r^3)} \]

- Kepler’s 3\textsuperscript{rd} Law when \( \Lambda \rightarrow 0 \).

Compare to the Kepler-like relation for a marble on the warped spandex fabric in the small curvature regime..

\[ T^3 \propto \frac{r^2}{(M + \pi \sigma_0 r^2)^{1/2}} \]

- Areal mass density, \( \sigma_0 \), plays the role of a negative cosmological constant, \( \Lambda \).

\*N. Cruz, M. Olivares, and J. Villanueva, “The geodesic structure of the Schwarzschild Anti-de Sitter black hole”, Classical and Quantum Gravity 22, 1167 (2005)
Conclusion

• The mass of the spandex fabric interior to the orbit of a marble matters.

• The modulus of elasticity, $E$, describing the spandex fabric is *not* constant and is a function of the stretch.

• Areal mass density, $\sigma_0$, plays the role of a *negative* cosmological constant, $\Lambda$. 