Circular orbits on a warped spandex fabric

Prof. Chad A. Middleton

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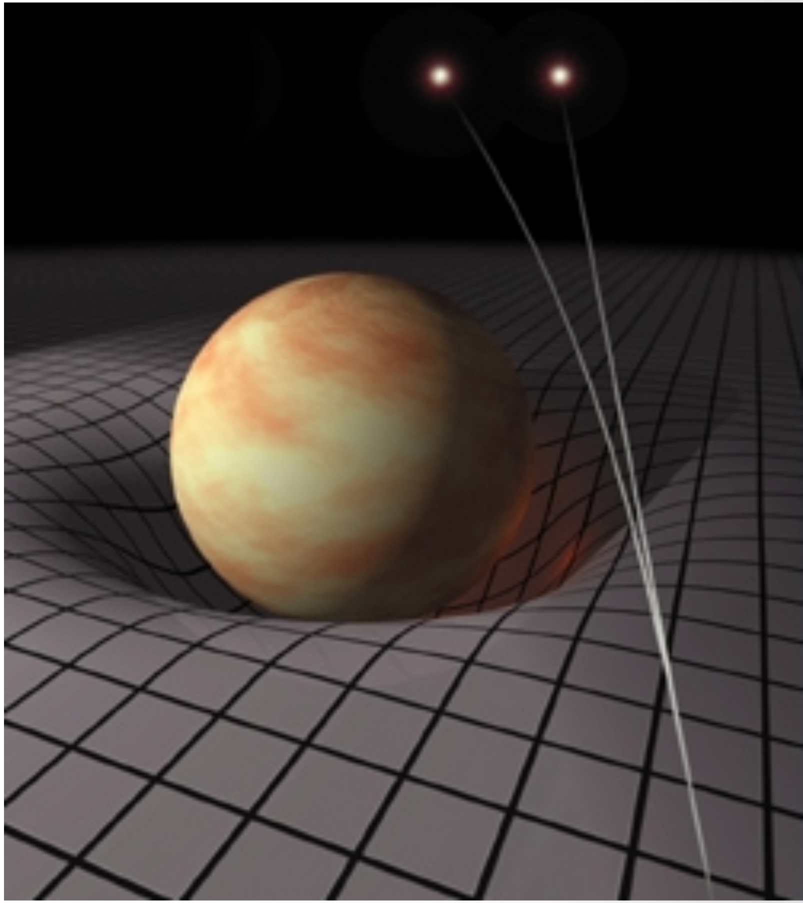
Einstein's theory of general relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $G_{\mu\nu}$ describes the *curvature of spacetime*
- $T_{\mu\nu}$ describes the *matter & energy in spacetime*

Matter tells *space* how to *curve*,
space tells *matter* how to *move*.



Sean M. Carrol, *Spacetime and Geometry: An Introduction to Einstein's General Relativity* (Addison Wesley, 2004)

Einstein's theory of general relativity

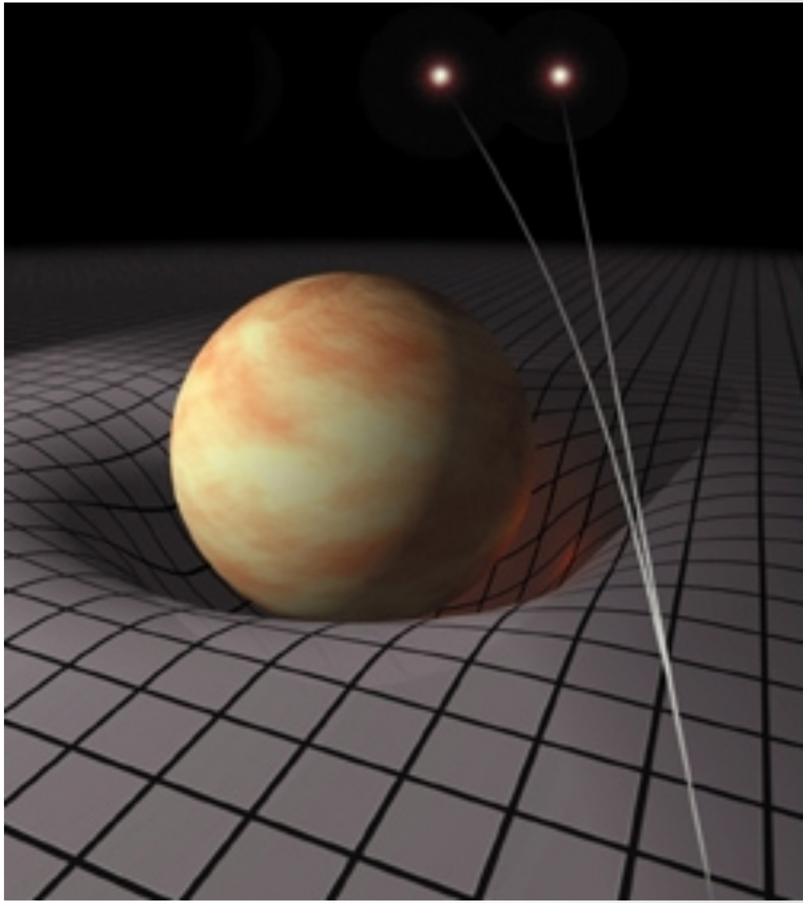
Consider a *spherically-symmetric, non-rotating massive object…*

Embedding diagram ($t = t_0$, $\theta = \pi/2$)..

- 2D equatorial 'slice' of the 3D space at one moment in time

$$z(r) = 2\sqrt{2M\,(r - 2M)}$$

Is there a warped 2D surface that will yield the orbits of planetary motion?



Sean M. Carrol, *Spacetime and Geometry: An Introduction to Einstein's General Relativity* (Addison Wesley, 2004)

Kepler's 3rd Law for planetary motion

- Newton's 2nd Law..

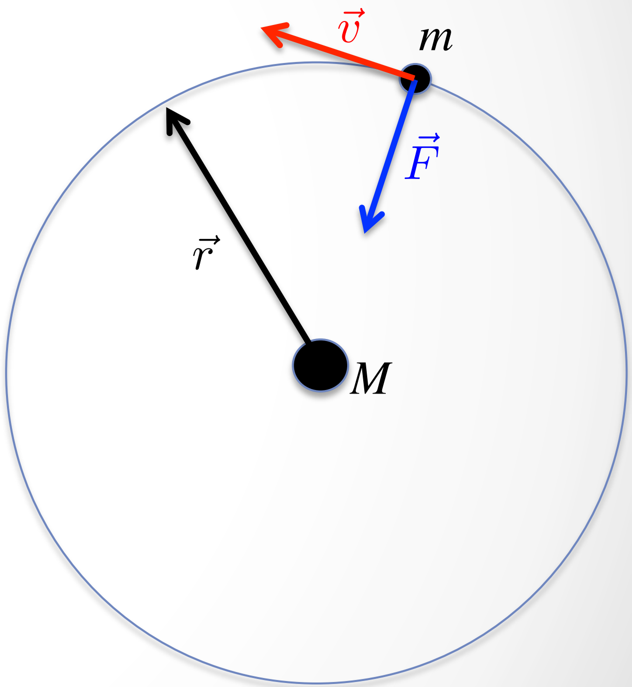
$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

- using the relation..

$$v = 2\pi r/T$$

- yields Kepler's 3rd Law..

$$T^2 = \left(\frac{4\pi^2}{G}\right) \cdot \frac{r^3}{M}$$



Notice:

- Kepler's 3rd Law is *independent* of *m*!

Outline

- A marble rolling on a *cylindrically-symmetric* surface
  (Lagrangian dynamics)
- The shape of the spandex fabric (Calculus of Variations)
- Small curvature regime
  - Kepler-like expression
  - Experimentation
- Large curvature regime
  - Kepler-like expression
  - Direct measurement of the *modulus of elasticity*
  - Experimentation
- Circular orbits in GR
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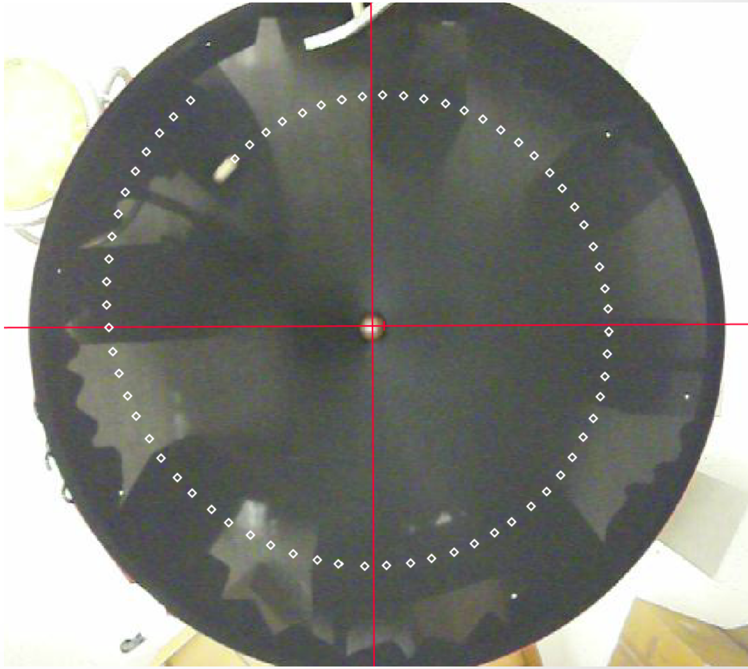


A marble rolling on a *cylindrically-symmetric* surface

- is described by a Lagrangian of the form..

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}I\omega^2 - mgz$$

- Now, for the marble..

$$I = \frac{2}{5}mR^2 \text{ and } \omega^2 = v^2/R^2 \qquad \text{so} \qquad \frac{1}{2}I\omega^2 = \frac{1}{5}mv^2$$

Notice:

- The marble is constrained to reside on the fabric..
- $$z = z(r)$$

The Lagrange equation of motion

for the radial-coordinate..

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

- yields the equation of motion for the *marble*..

$$(1 + z'^2)\ddot{r} + z'z''\dot{r}^2 - r\dot{\phi}^2 + \frac{5}{7}gz' = 0 \quad *$$

- compare to the equation of motion for *planetary orbits*..

$$\ddot{r} - r\dot{\phi}^2 + \frac{GM}{r^2} = 0$$

* will NOT yield *Newtonian-like orbits of planetary motion* for a marble on ANY cylindrically-symmetric surface!

L.Q. English and A. Mareno, "Trajectories of rolling marbles on various funnels", Am. J. Phys. **80** (11), 996-1000 (2012)

Circular orbits on a cylindrically-symmetric surface

For the equation of motion for the marble..

$$(1 + z'^2)\ddot{r} + z'z''\dot{r}^2 - r\dot{\phi}^2 + \frac{5}{7}gz' = 0$$

- setting $\dot{r} = \ddot{r} = 0$ for circular orbits, we obtain..

$$\frac{4\pi^2 r}{T^2} = \frac{5}{7}g \cdot z'(r)$$

Notice:

- we used the relation $v = r\dot{\phi} = 2\pi r/T$
- depends *linearly* on the *slope* of the spandex fabric.

The shape of the spandex fabric

Technique:

1. Construct potential energy (PE) *integral functional* of spandex fabric.
    i. *Elastic* PE of the *spandex*.
    ii. *Gravitational* PE of the *spandex*.
    iii. *Gravitational* PE of the *central mass*.

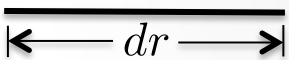
2. Apply *Calculus of Variations*.
    ⇒ The elastic fabric-mass system will assume the shape which *minimizes* the *total* PE of the system.

i. and *iii*. first considered in "Comment on "The shape of 'the Spandex' and orbits upon its surface", Am. J. Phys. 70 (10), 1056-1058 (2002)

Elastic PE of the *spandex fabric*

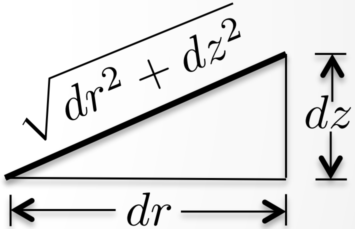
Elastic PE of a *differential concentric ring* of the fabric of unstretched width *dr*…

$$dU_e = \frac{1}{2}\kappa(\sqrt{dr^2 + dz^2} - dr)^2$$



- Define the *modulus of elasticity*, *E* ..

$$E = \frac{\kappa dr}{2\pi r}$$



- Integrating the differential segment over the whole fabric, the *total* elastic PE of the fabric is…

$$U_e = \int_0^R \pi E \cdot r(\sqrt{1 + z'^2} - 1)^2 \, dr$$

Gravitational PE of the *spandex fabric*

Gravitational PE of a *differential concentric ring* of the fabric..

$$dU_{g,s} = dm_s g \cdot z$$

- The mass of the differential ring is a constant under stretching..

$$dm_s = \sigma_0 \cdot 2\pi r dr = \sigma(z') \cdot 2\pi r \sqrt{dr^2 + dz^2}$$

where $\sigma_0, \sigma(z')$ are the *unstretched*, *variable* areal mass densities.

- Integrating the differential segment over the whole fabric, the *total* gravitational PE of the fabric is..

$$U_{g,s} = \int_0^R 2\pi\sigma_0 g \cdot rz \, dr$$

Gravitational PE of the *central mass*

$$U_{g,M} = Mg \cdot z(0) = -\int_0^R Mg \cdot z'(r)\, dr$$

Notice:

- we approximate the central mass as being point-like.

The *total* PE of the spandex-central mass system

$$U = U_e + U_{g,s} + U_{g,M} = \int_0^R f(z, z'; r)\, dr$$

where we defined the functional..

$$f(z, z'; r) \equiv \pi E \cdot r(\sqrt{1 + z'^2} - 1)^2 + 2\pi\sigma_0 g \cdot rz - Mg \cdot z'$$

To *minimize* the *total* PE, subject to the Euler-Lagrange eqn..

$$\frac{\partial f}{\partial z} - \frac{d}{dr}\frac{\partial f}{\partial z'} = 0$$

The shape equation for the elastic fabric

The Euler-Lagrange equation takes the form..

$$\frac{d}{dr}\left[rz'\left[1-\frac{1}{\sqrt{1+z'^2}}\right]-\frac{Mg}{2\pi E}\right]=\frac{\sigma_0 g}{E}\cdot r$$

- which can be integrated..

$$rz'\left[1-\frac{1}{\sqrt{1+z'^2}}\right]=\alpha(M+\pi\sigma_0 r^2)$$

- where we defined the parameter..

$$\alpha\equiv\frac{g}{2\pi E}$$

The circular equation of motion & the shape equation

$$\frac{4\pi^2 r}{T^2} = \frac{5}{7} g \cdot z'(r)$$

$$rz' \left[1 - \frac{1}{\sqrt{1+z'^2}} \right] = \alpha(M + \pi\sigma_0 r^2)$$

- Circular equation of motion
- The shape equation

- Small curvature regime..

$$z'(r) \ll 1 \quad \text{so} \quad \frac{1}{\sqrt{1+z'^2}} = 1 - \frac{1}{2} z'^2 + o(z'^4)$$

- Large curvature regime..

$$z'(r) \gg 1 \quad \text{so} \quad \frac{1}{\sqrt{1+z'^2}} = \frac{1}{z'} \frac{1}{\sqrt{1+1/z'^2}} = \frac{1}{z'}\left(1 - \frac{1}{2z'^2} + o(1/z'^4)\right)$$

The small curvature regime

When $z'(r) \ll 1$ ...

- expanding the shape equation and inserting into the circular eqn of motion

$$T^3 = \left(\frac{28\pi^2}{5g}\right)^{3/2} \frac{1}{\sqrt{2\alpha}} \cdot \frac{r^2}{(M + \pi\sigma_0 r^2)^{1/2}}$$

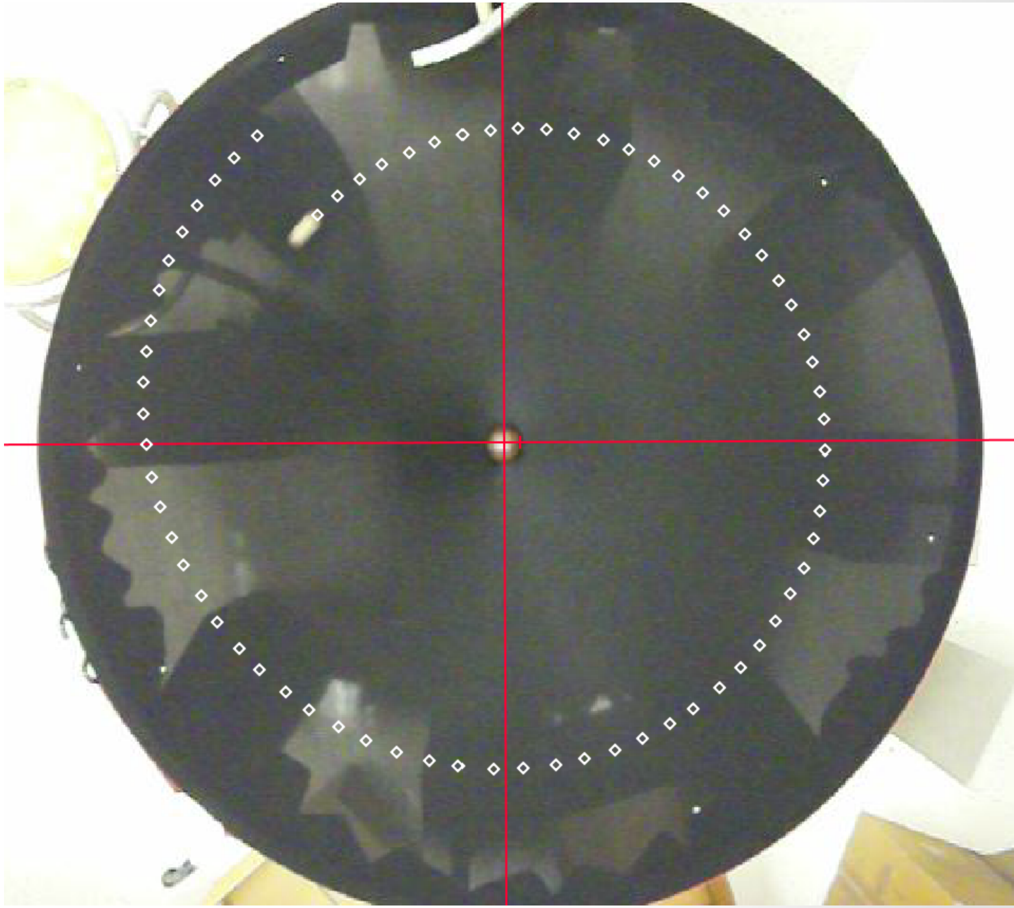
Notice:

- $T^3 \propto r^2/\sqrt{M}$ * when $M \gg \pi\sigma_0 r^2$
- $T^3 \propto r/\sqrt{\sigma_0}$ when $M \ll \pi\sigma_0 r^2$
- Two competing terms on equal footing when $M \simeq \pi\sigma_0 r^2 \sim 0.10$ kg

*Gary D. White and Michael Walker, "The shape of 'the Spandex' and orbits upon its surface", Am. J. Phys. 70 (1), 48-52 (2002).

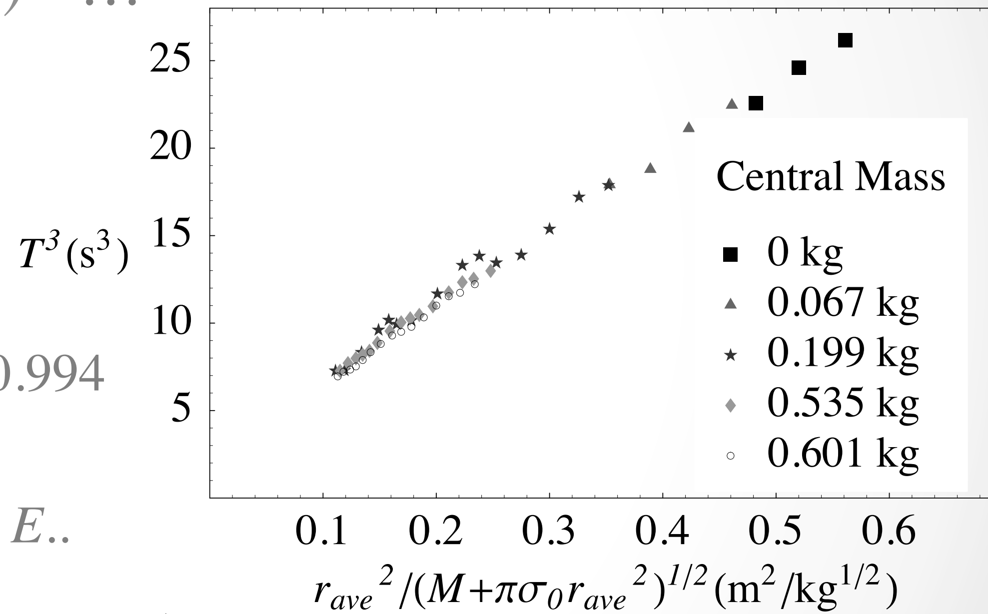
The experiment in the small curvature regime

- 4 ft. diameter trampoline frame
  - styrofoam insert for *zero* pre-stretch
  - truck tie down around perimeter
- Camera mounted directly above, ramp mounted on frame
- Most *circular* video clip (of ~12) imported into Tracker
- Position determined every 1/30 s and *average radius*, r_{ave}, calculated per revolution
- Shift by 1/8 revolution for subsequent data point



The experiment in the small curvature regime

Plot of T^3 vs $r_{ave}{}^2/(M+\pi\sigma_0 r_{ave}{}^2)^{1/2}$…



Notice:

- orbit for *zero* central mass!
- slope = 45.6 kg$^{1/2}$ s^3/m^2 w/ $R^2 = 0.994$

The *slope* yields a value for α and E..

$$\alpha = \frac{1}{2}\left(\frac{28\pi^2}{5g}\right)^3 \cdot \frac{1}{\text{slope}^2} \simeq 0.043 \text{ m/kg}$$

$$E = \frac{g}{2\pi\alpha} \simeq 36 \text{ N/m}$$

The large curvature regime

When $z'(r) \gg 1$ ...

- expanding the shape equation and inserting into the circular eqn of motion

$$T = \left(\frac{28\pi^2}{5g}\right)^{1/2} \cdot \frac{r}{(M\alpha + r)^{1/2}}$$

Notice:

- $T \propto r/\sqrt{M}$ * when $r \ll M\alpha$
- When $r \gg M\alpha$, $z'(r) \simeq 1$, so above equation *invalid*!
- Using $\alpha = 0.043$ m/kg , we get *poor* results!

* corresponds to the solution of the 2D Laplace equation with cylindrical-symmetry

Direct measurement of the *modulus of elasticity, E*

The *shape equation* in the large curvature regime is..

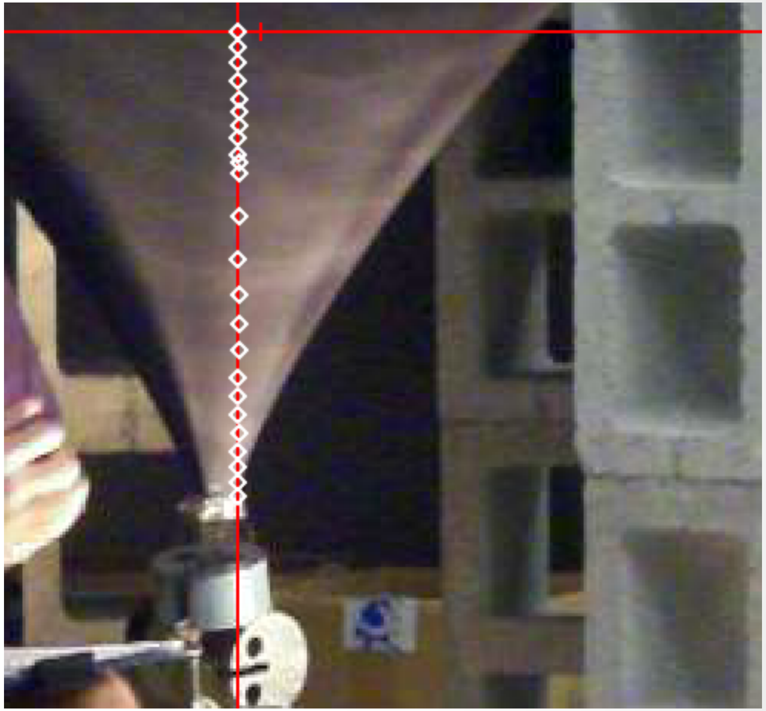
$$z'(r) \simeq \frac{M\alpha}{r} + 1$$

- integrating yields..

$$\frac{z(M)}{\ln(R_B)} = (M - M_0)\alpha$$

Notice:

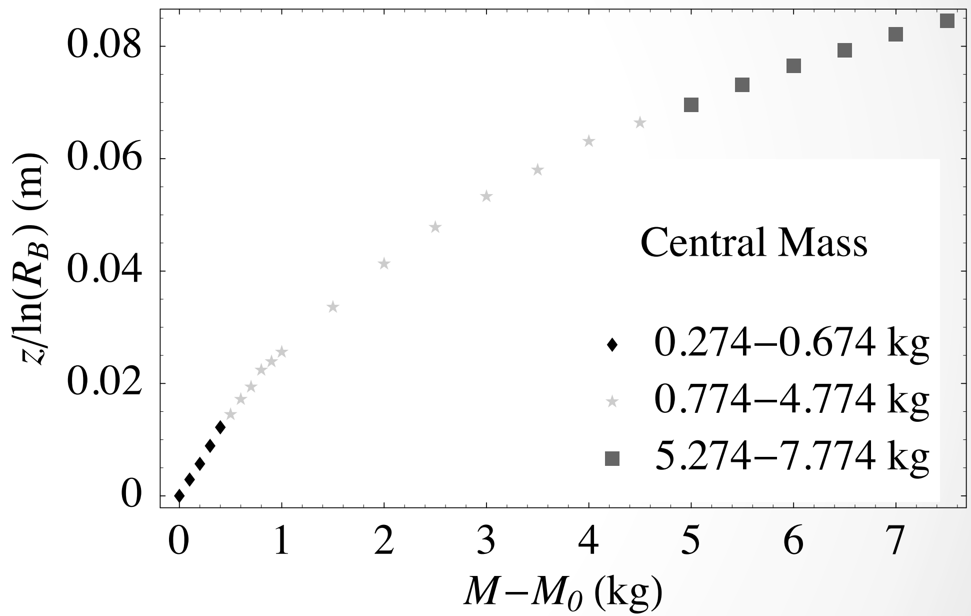
- Plot of $z(M)/\ln(R_B)$ vs $(M - M_0)$ yields the *slope*, which ***is*** the value of α!



Top 10 diamonds:
M = 0.274kg - 1.174kg in 0.1 kg intervals
Bottom 14 diamonds:
M = 1.274kg - 7.774kg in 0.5 kg intervals

Direct measurement of the *modulus of elasticity, E*

Plot of $z(M)/\ln(R_B)$ vs $(M - M_0)$ yields the *slope*, which ***is*** the value of α!



Notice:

- $M = 0.274 - 0.674$ kg regime

$$\alpha \simeq 0.030 \text{ m/kg}$$

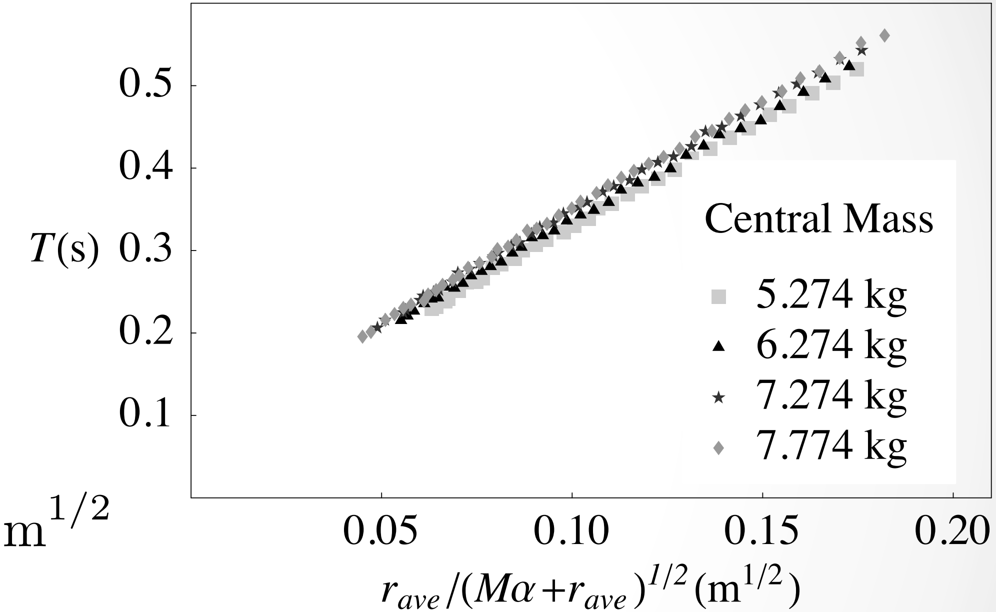
- $M = 5.274 - 7.774$ kg regime

$$\alpha \simeq 0.006 \text{ m/kg}$$

The experiment in the large curvature regime

Plot of T vs $r_{ave}/(M\alpha+r_{ave})^{1/2}$..

- $\text{slope}_{exp} = 2.62\ \text{s/m}^{1/2}$
- $\text{slope}_{th} = \left(\frac{28\pi^2}{5g}\right)^{1/2} = 2.37\ \text{s/m}^{1/2}$



Notice:

- ~10% error for α = 0.006 kg/m, which compares
- with ~94% error when α = 0.043 kg/m.

Circular orbits in GR

The metric exterior to a *spherically-symmetric* object of *mass M*,
in the presence of a *cosmological constant* (or *vacuum energy*), Λ..

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where

$$\rho_{vac} = \frac{\Lambda}{8\pi G}$$

Notice..

- $\Lambda > 0$, Schwarzschild – de Sitter spacetime
- $\Lambda < 0$, Schwarzschild – Anti-de Sitter spacetime
- $\Lambda = 0$, Schwarzschild solution

Circular orbits in GR

By normalizing the four-velocity and employing conservation of energy and angular momentum..

- The radial equation of motion..

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \text{ where } V_{eff}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2)$$

For circular orbits, set..

- $\frac{d}{dr}V_{eff}(r) = 0$
- $\mathcal{E} = V_{eff}(r)$

Circular orbits in GR

One arrives at an *exact* Kepler-like expression of the form..

$$T^2 \propto \frac{r^3}{(M - \frac{\Lambda}{3G} r^3)} \quad ^*$$

- Kepler's 3rd Law when Λ -> 0.

Compare to the Kepler-like relation for a marble on the warped spandex fabric in the small curvature regime..

$$T^3 \propto \frac{r^2}{(M + \pi\sigma_0 r^2)^{1/2}}$$

- Areal mass density, σ_0, plays the role of a *negative* cosmological constant, Λ.

*N. Cruz, M. Olivares, and J. Villanueva, "The geodesic structure of the Schwarzschild Anti-de Sitter black hole", Classical and Quantum Gravity **22**, 1167 (2005)

Conclusion

- The mass of the spandex fabric interior to the orbit of a marble matters.
- The *modulus of elasticity, E,* describing the spandex fabric is *not* constant and is a function of the stretch.
- Areal mass density, σ_0, plays the role of a *negative* cosmological constant, Λ.