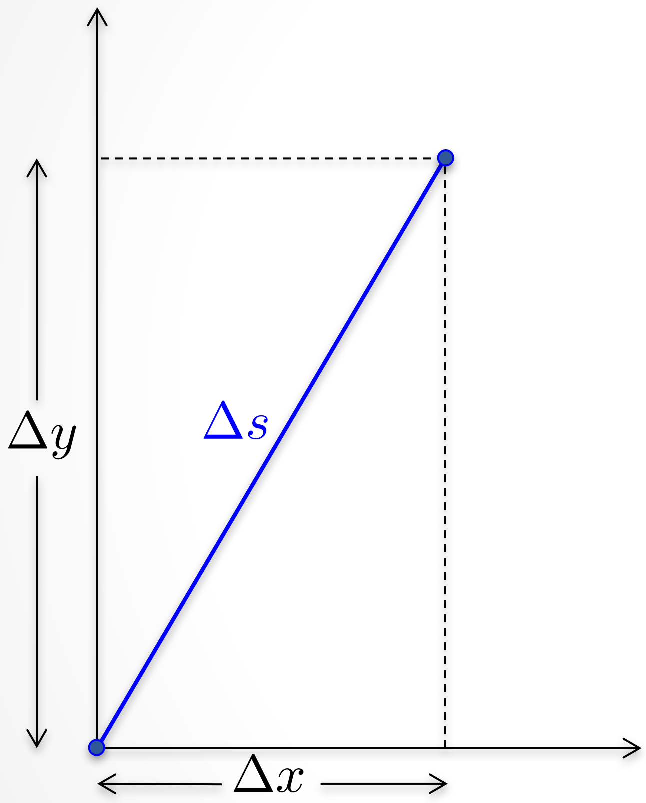
Einstein's Classical Theory of General Relativity and Black Holes

Prof. Chad A. Middleton
Physics Seminar
Colorado Mesa University
March 2, 2017

Outline…

- Length in 2D space
- Special relativity & the relativity of simultaneity
- ‘Length’ in 4D spacetime
- Spacetime diagrams & light cones
- The Schwarzschild solution & black holes

2D Euclidean Space



Line element in Euclidean space...

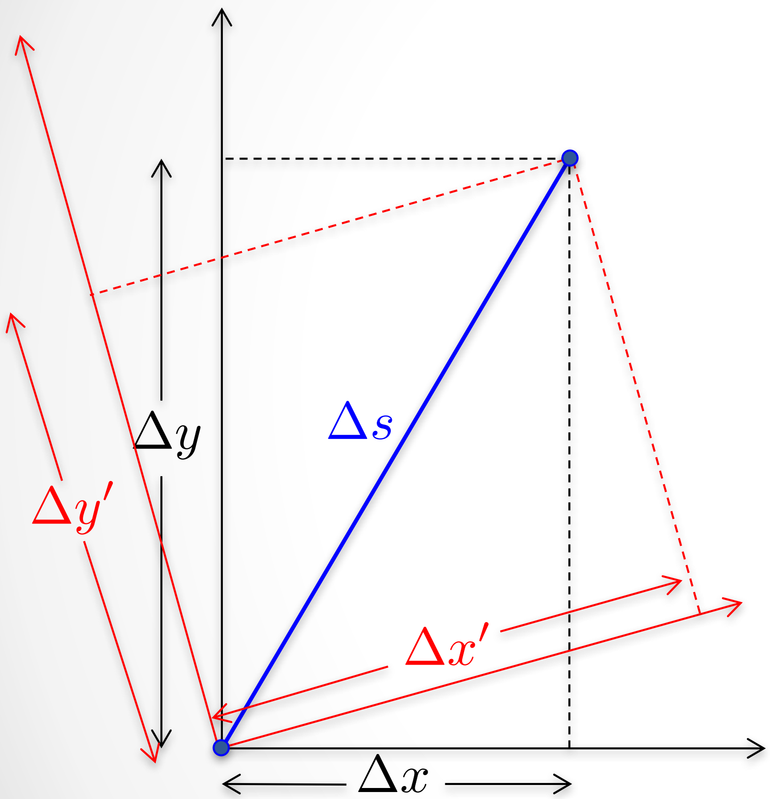
$$\lim_{\Delta \to 0} \Delta s^2 = \Delta x^2 + \Delta y^2$$

or

$$ds^2 = dx^2 + dy^2$$

- ds^2 is the *line element* measuring length

Rotations in 2D Euclidean Space



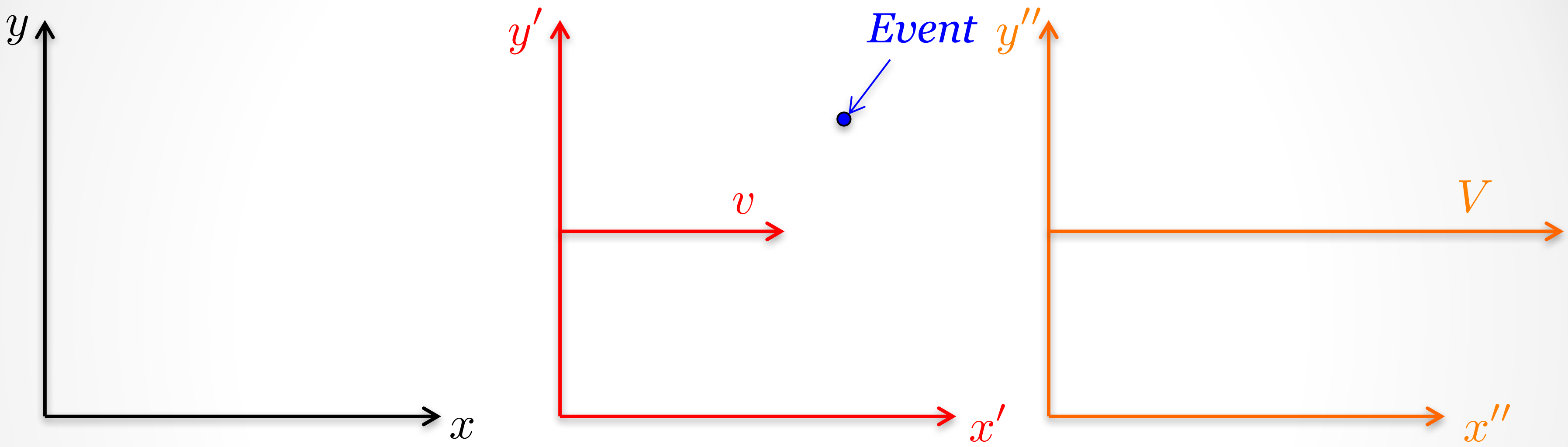
Line element in Euclidean space...

$$ds^2 = dx^2 + dy^2$$

$$= dx'^2 + dy'^2 = ds'^2$$

- ds^2 is the *line element* measuring length
- ds^2 is *invariant* under rotations

Einstein's theory of special relativity



Lorentz Boosts:

$$ct' = \gamma \left(ct - \frac{v}{c} x \right)$$

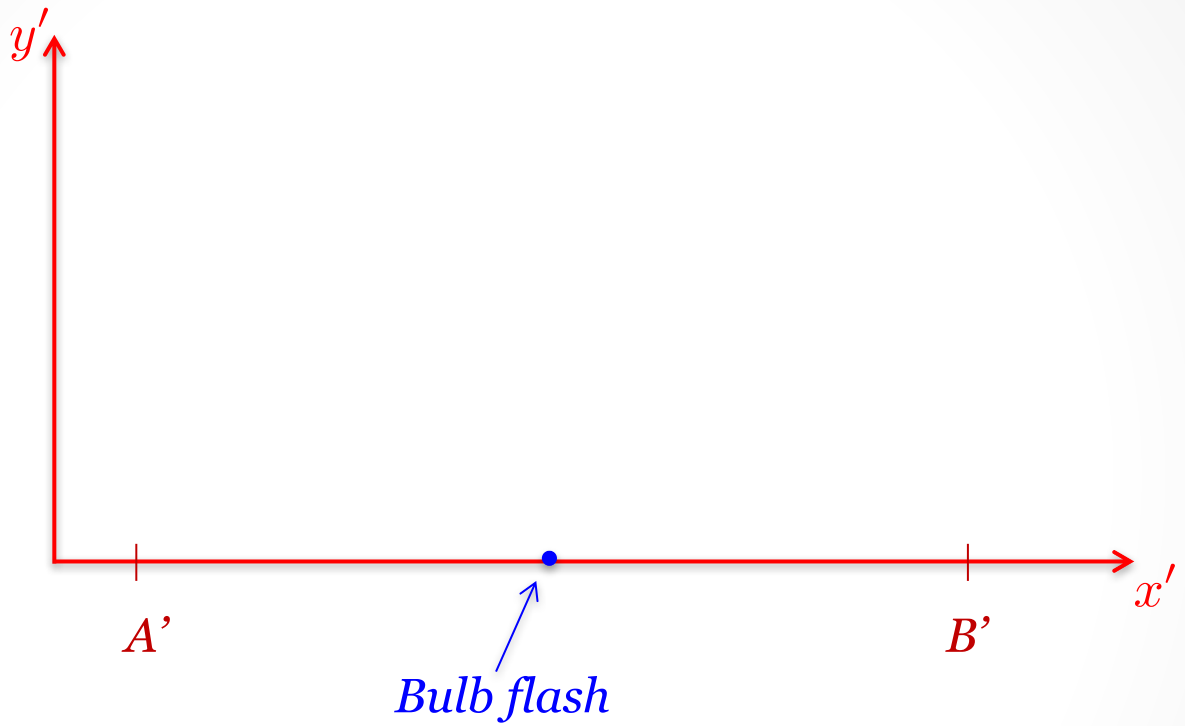
$$x' = \gamma \left(x - vt \right)$$

$$y' = y$$

$$z' = z$$

Time and 3D *length* depend on the frame of reference!

Relativity of Simultaneity



Notice:

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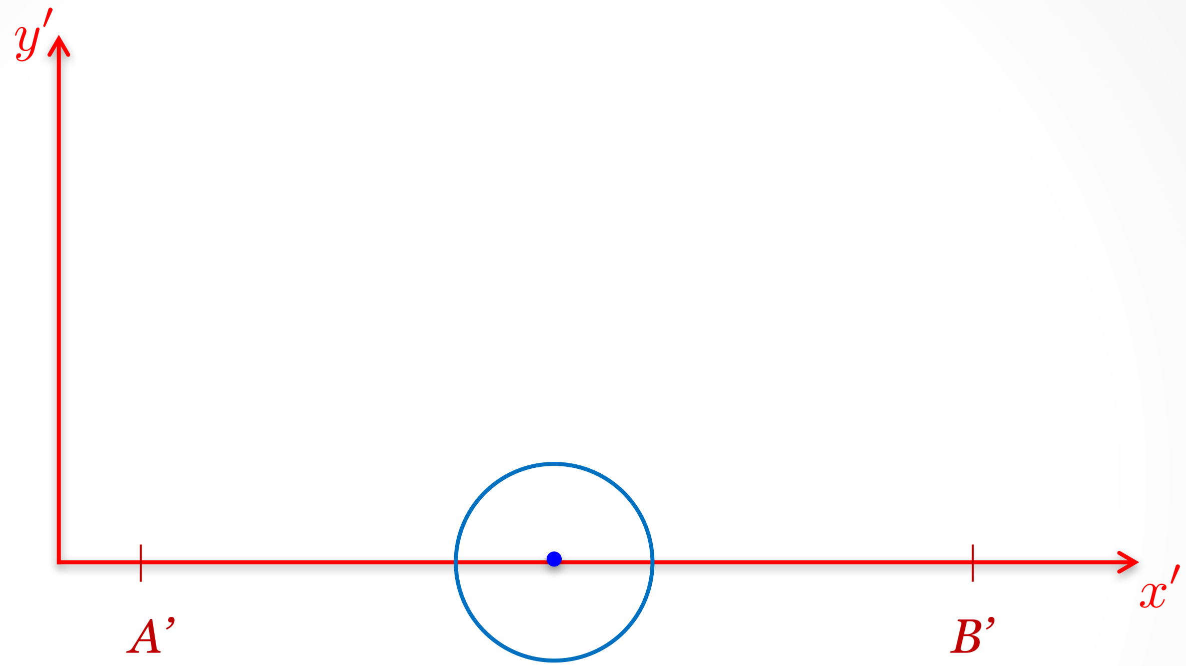
Relativity of Simultaneity



Notice:

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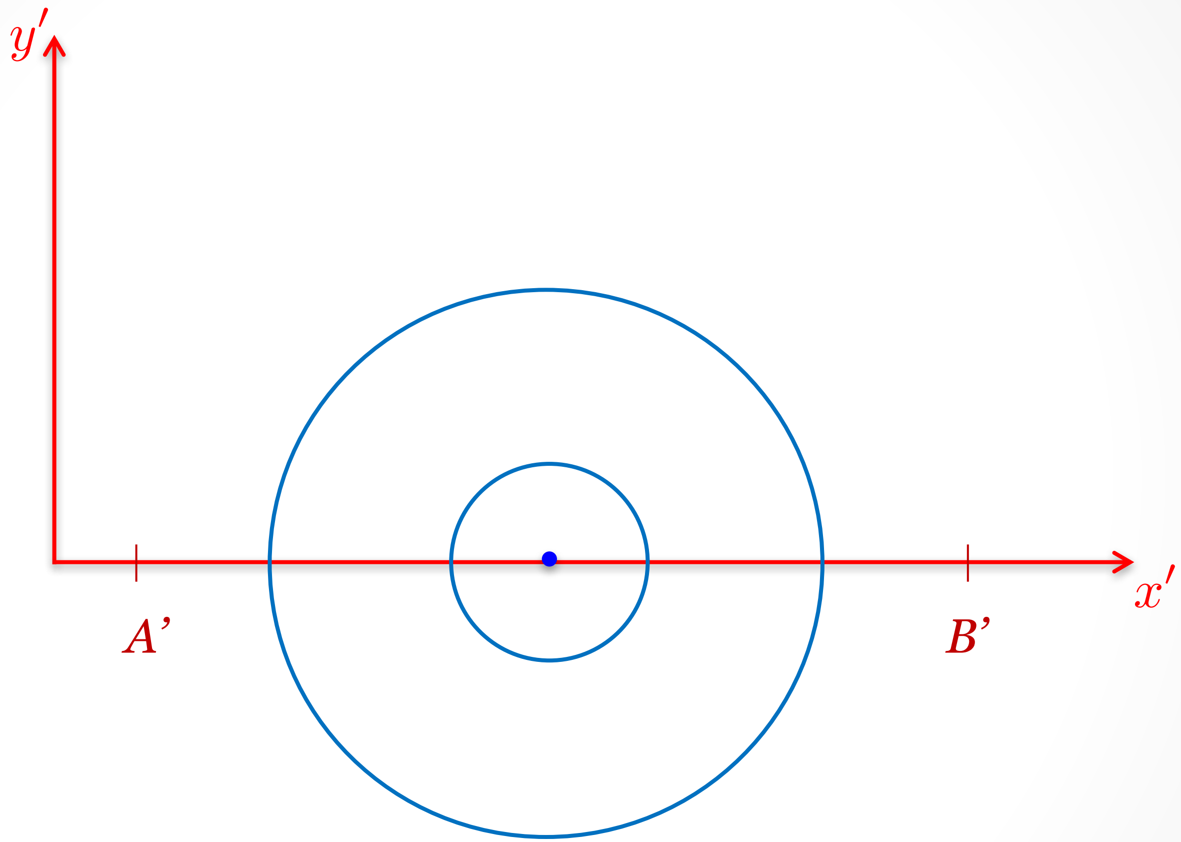

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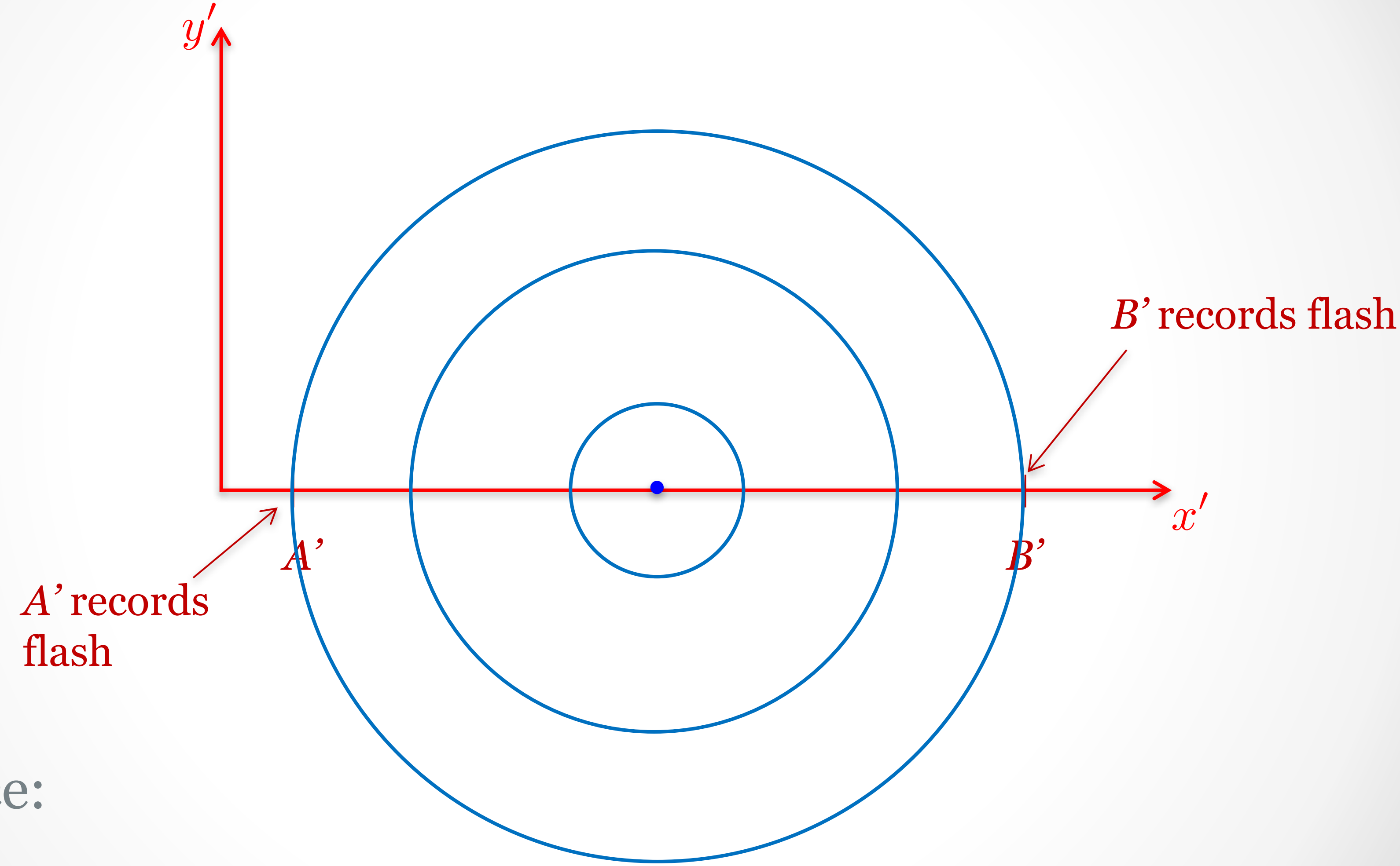

Notice:

-
- $t'_B = t'_A$
-

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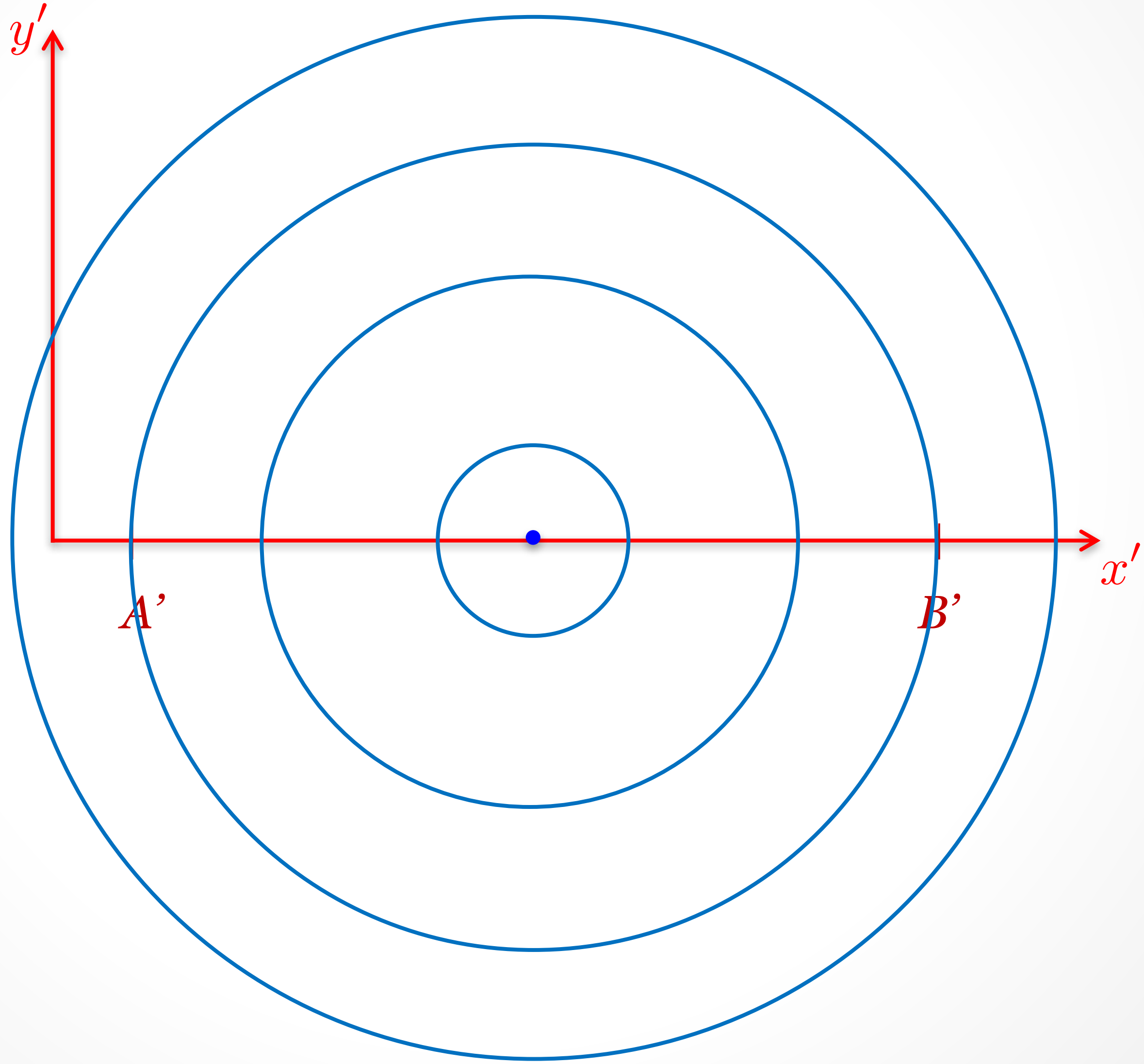

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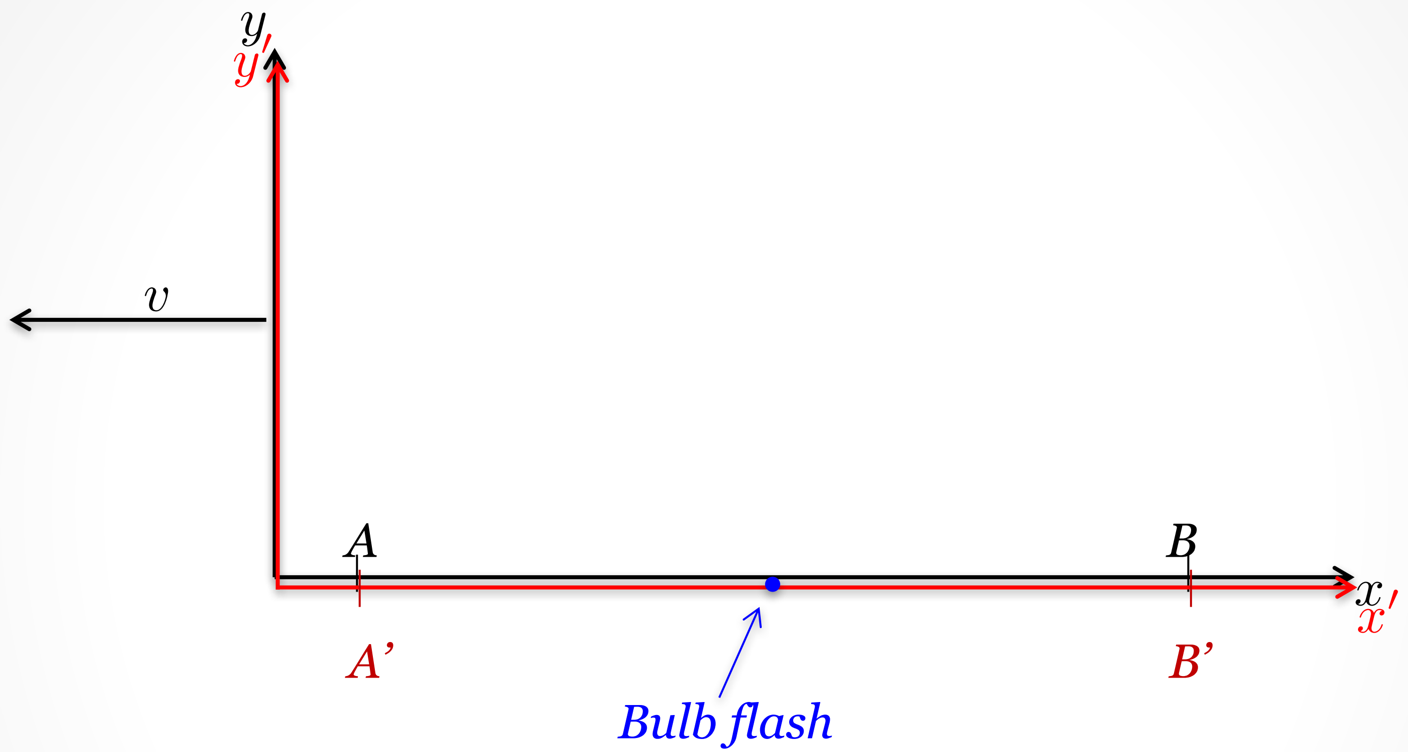

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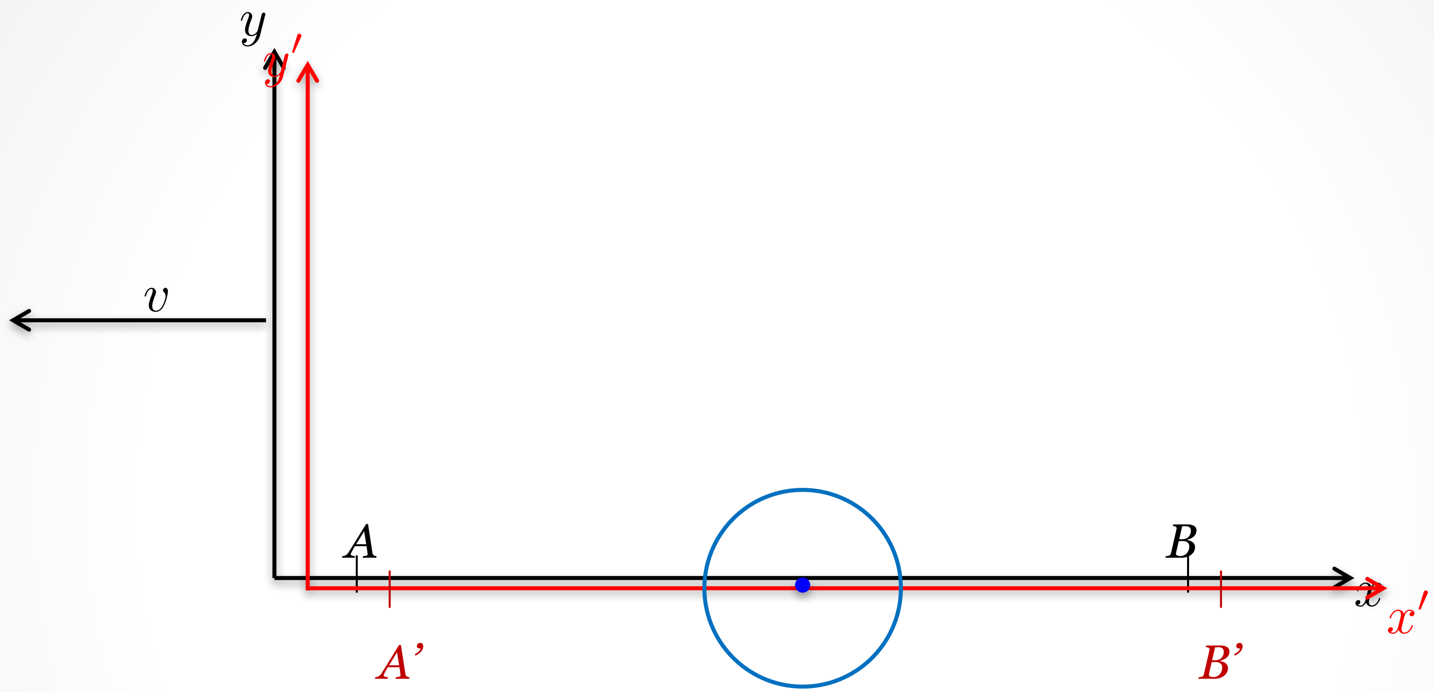

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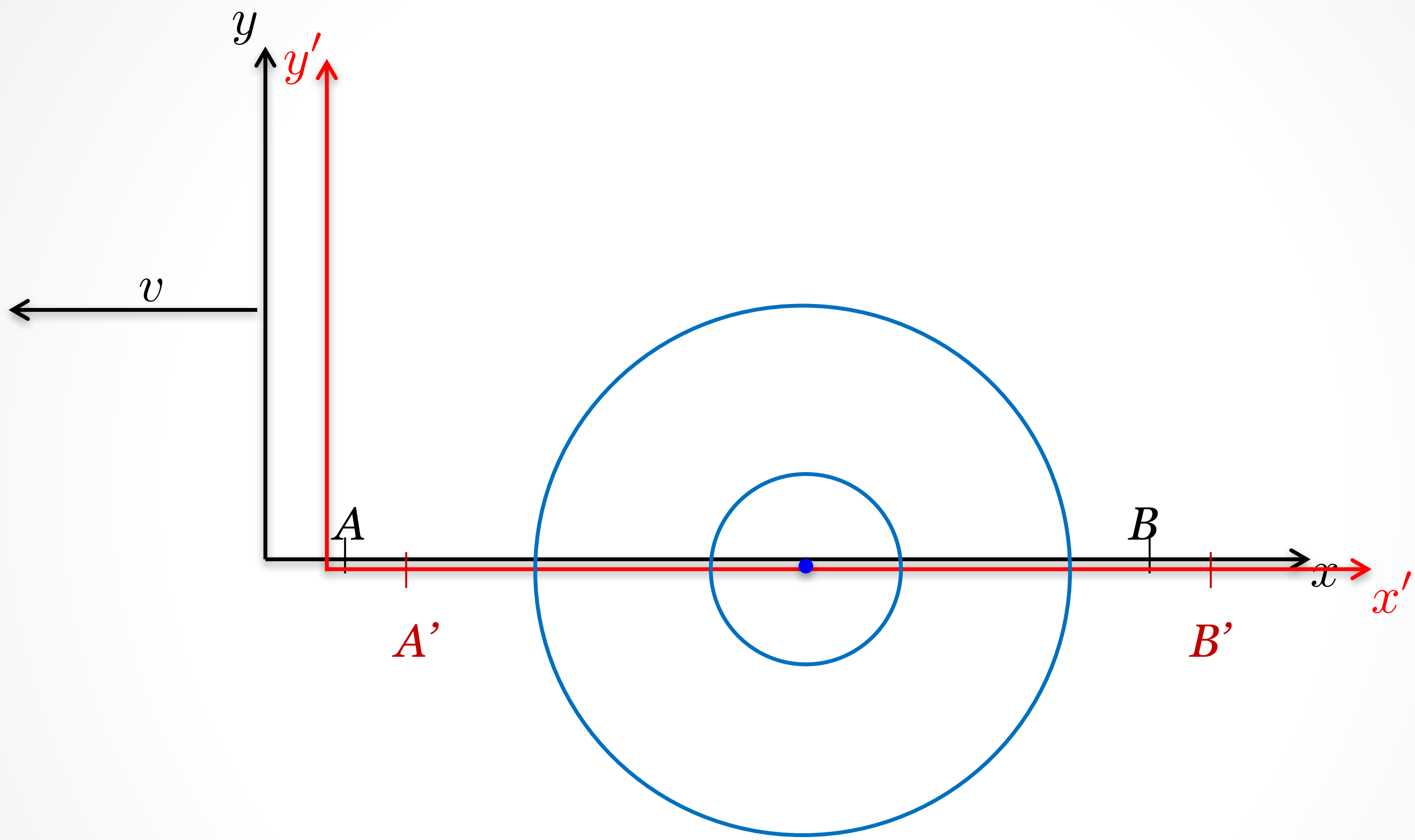

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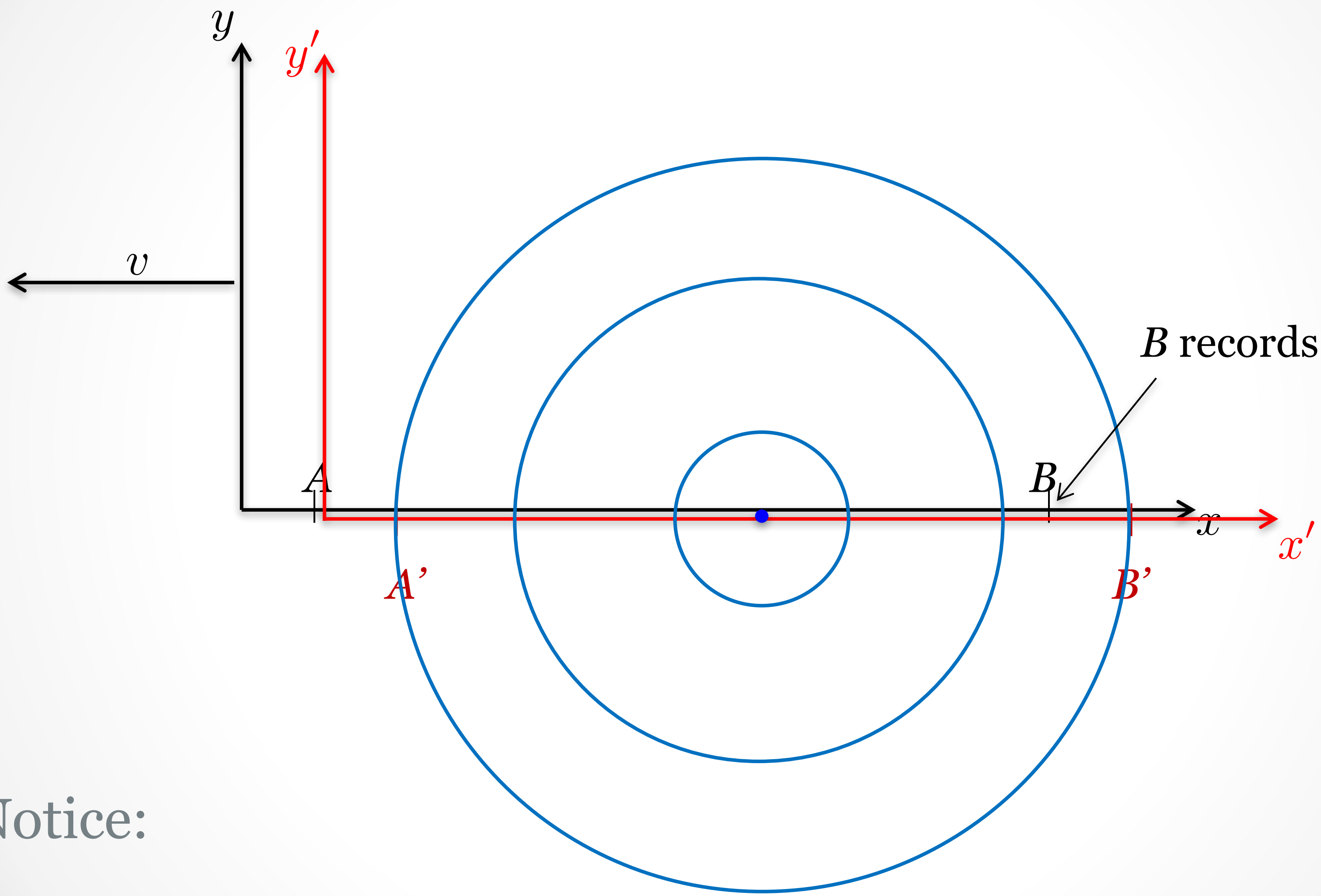

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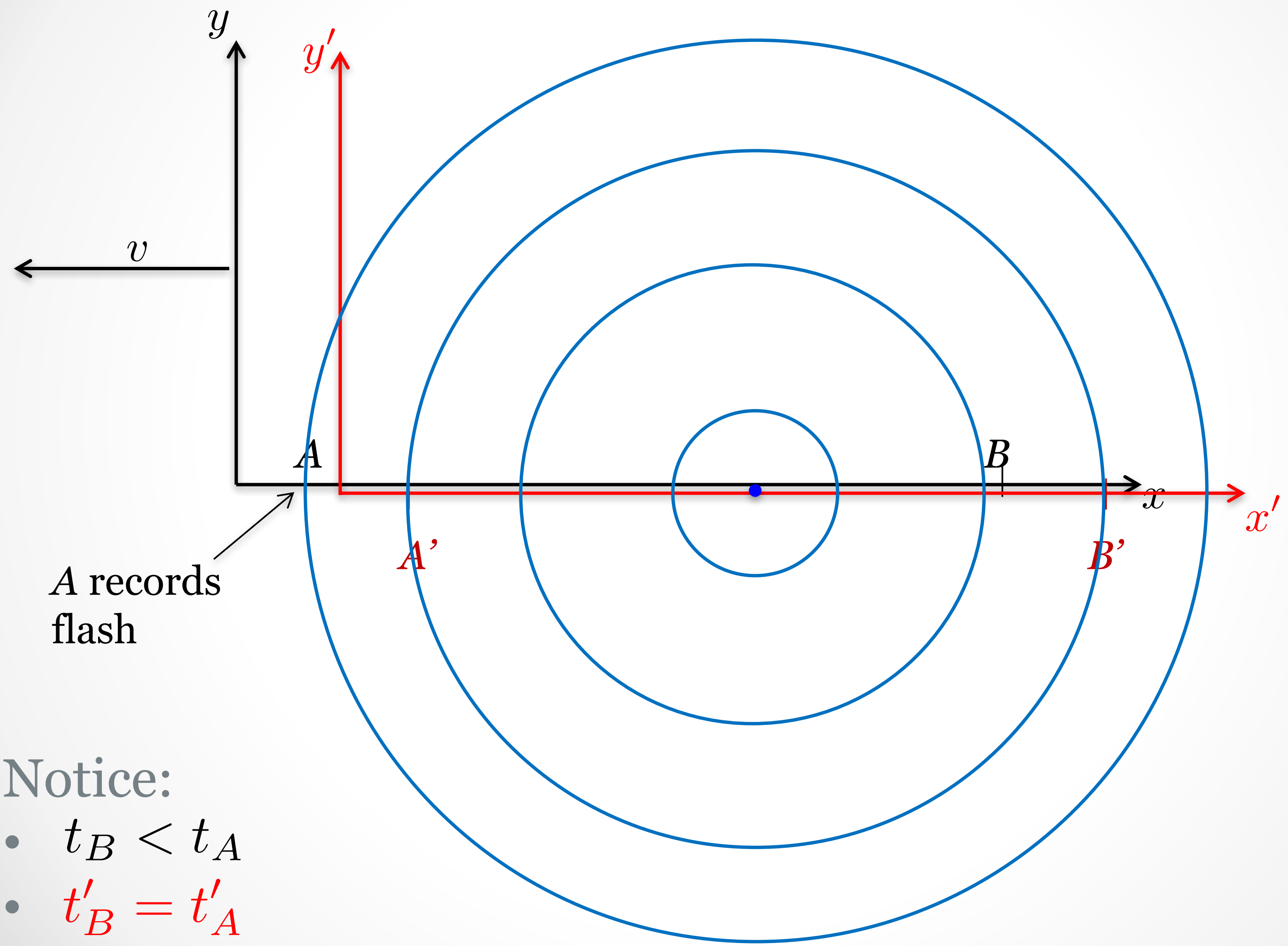

Notice:

- $t_B < t_A$
- $t'_B = t'_A$
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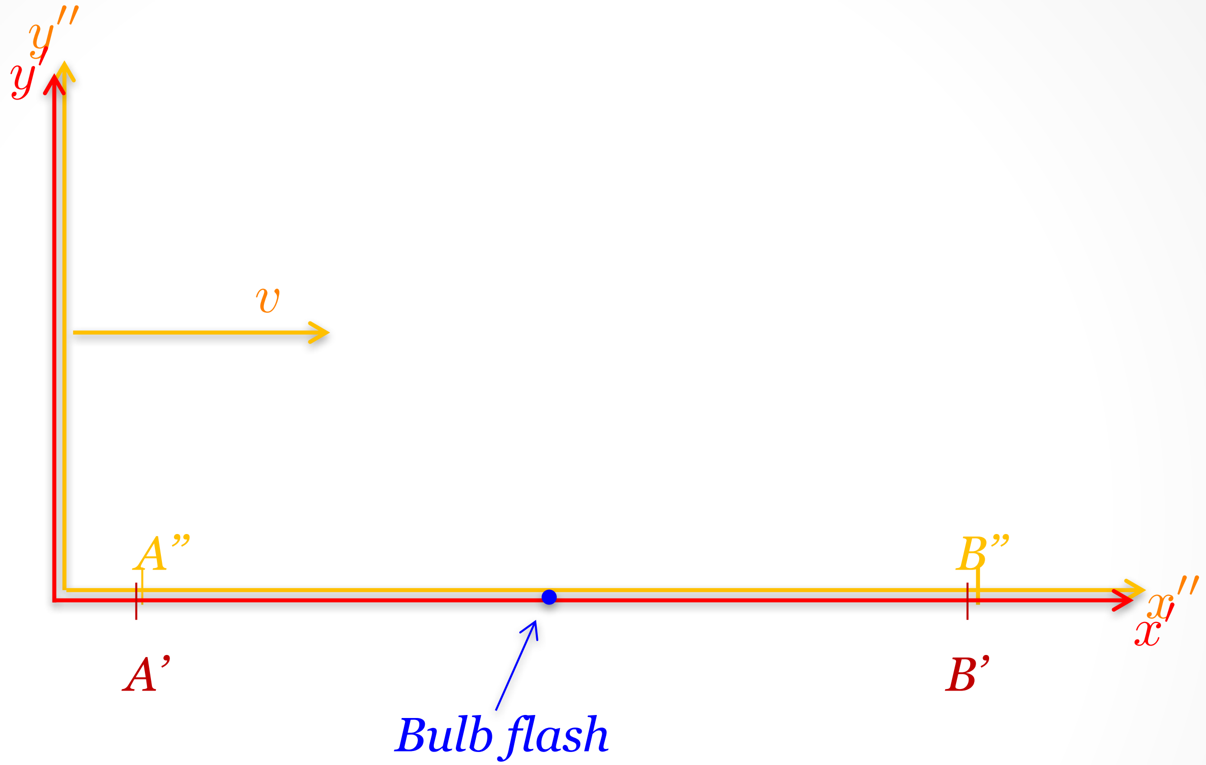

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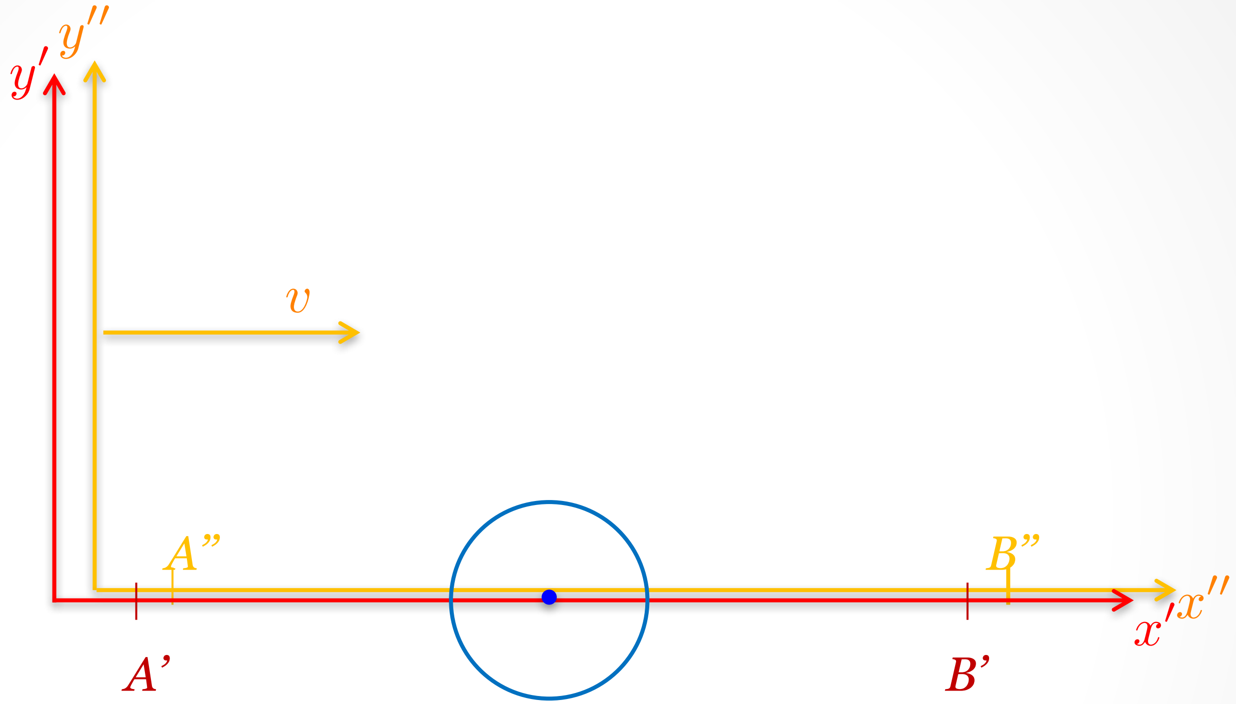

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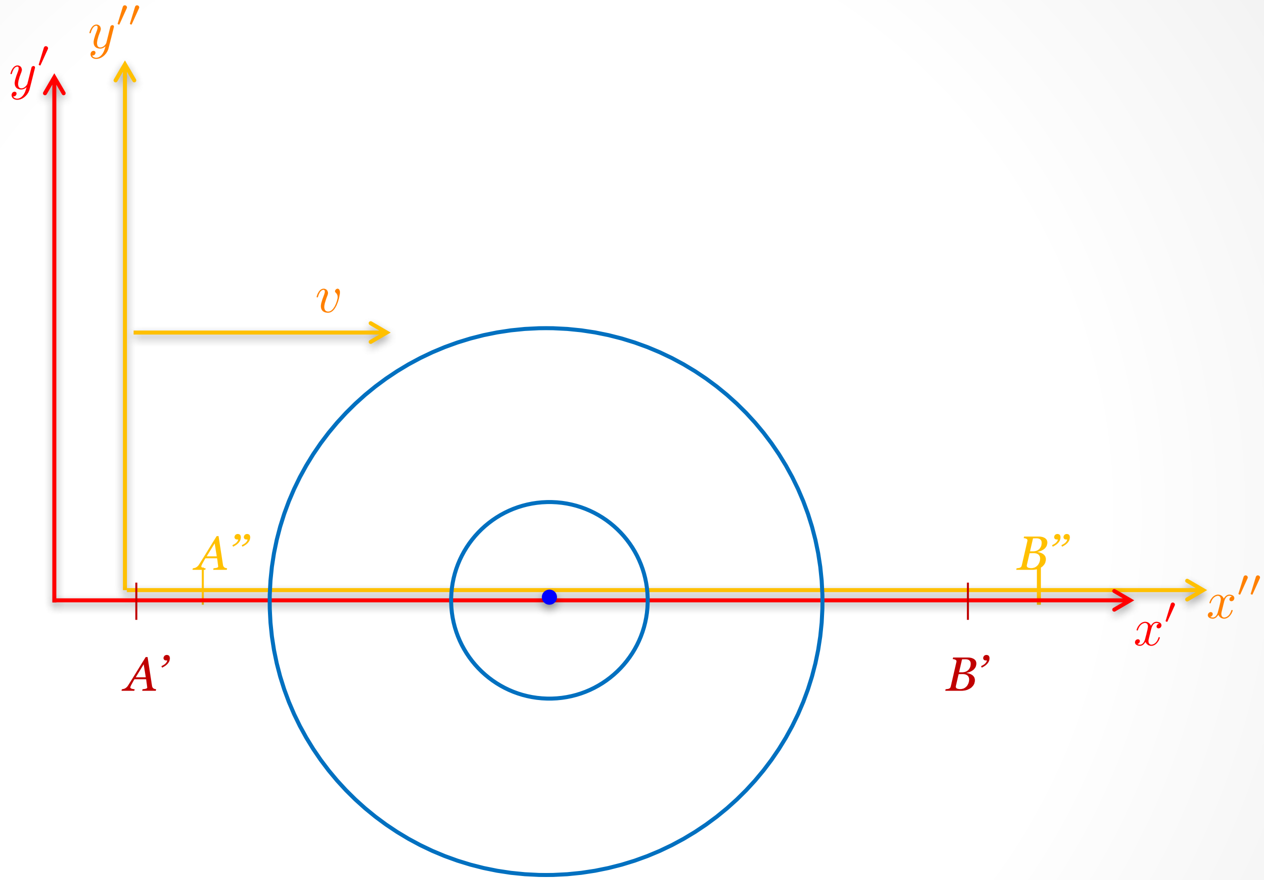

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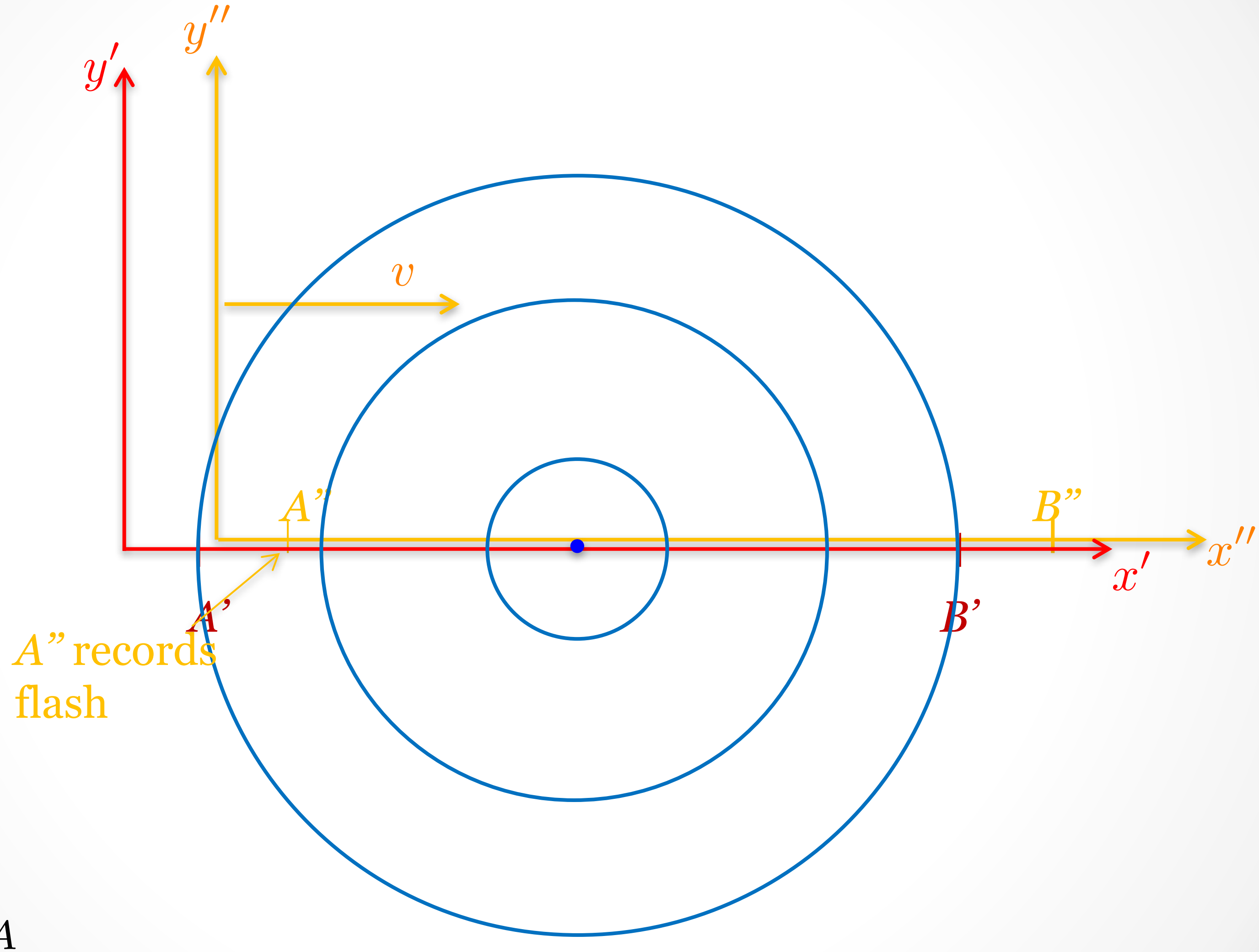

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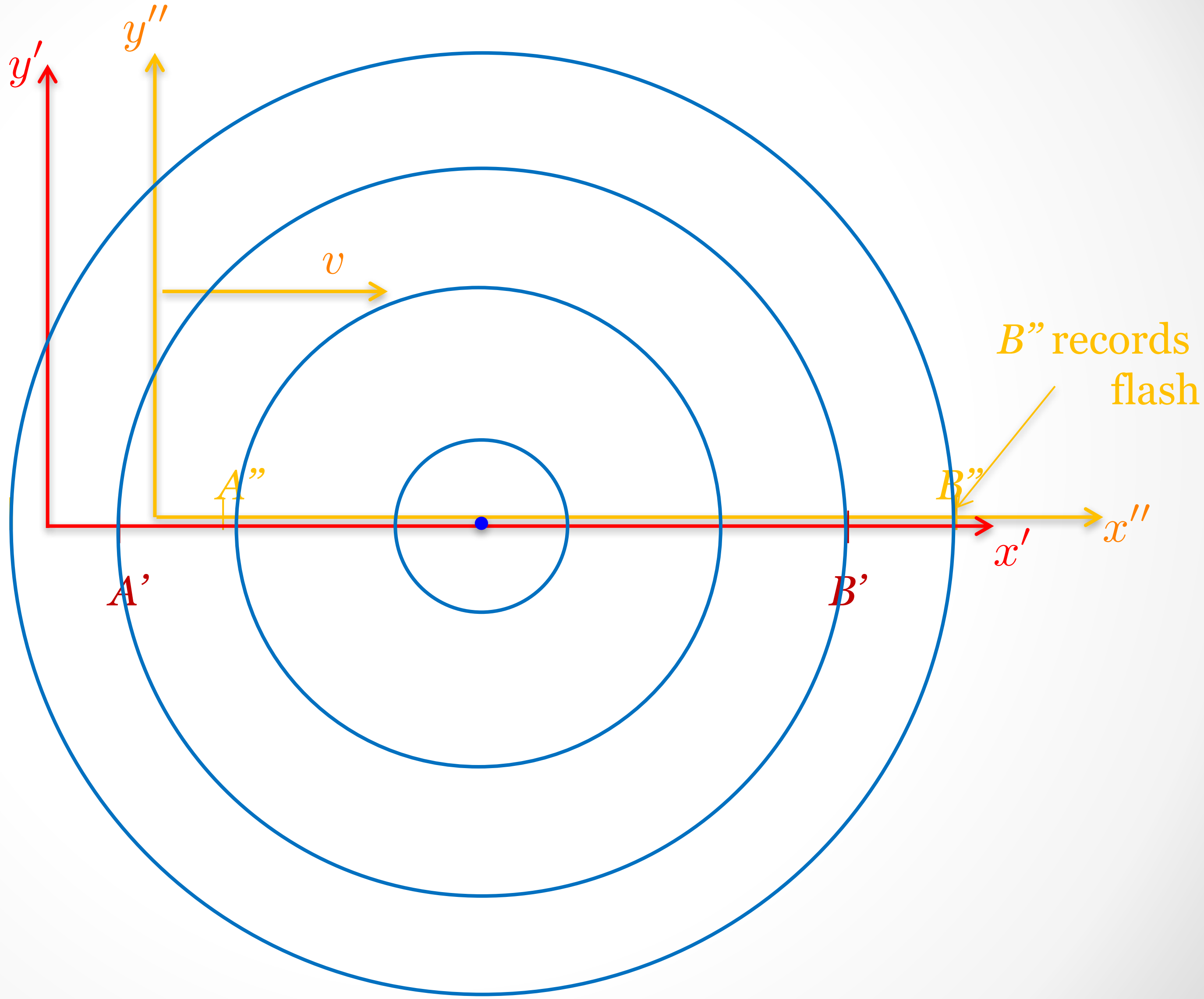
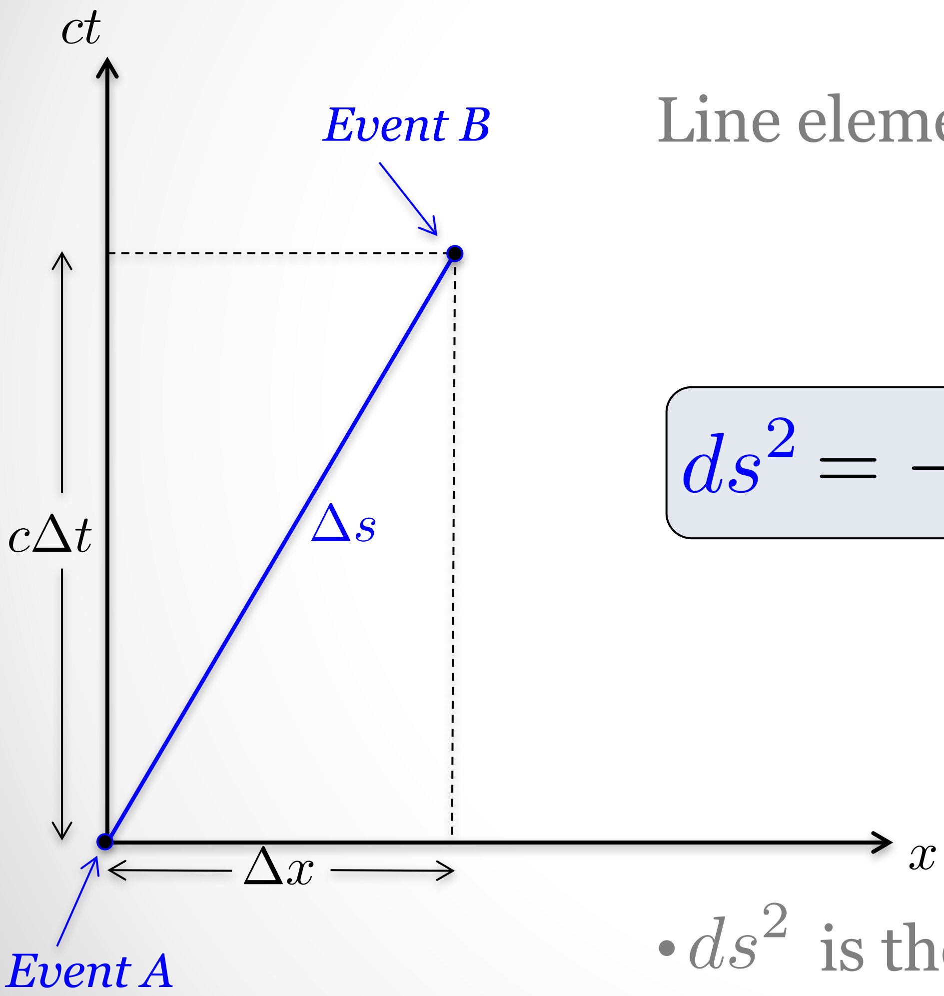


Notice:

- $t_B < t_A$
- $t'_B = t'_A$
- $t''_B > t''_A$

4D Minkowski Spacetime

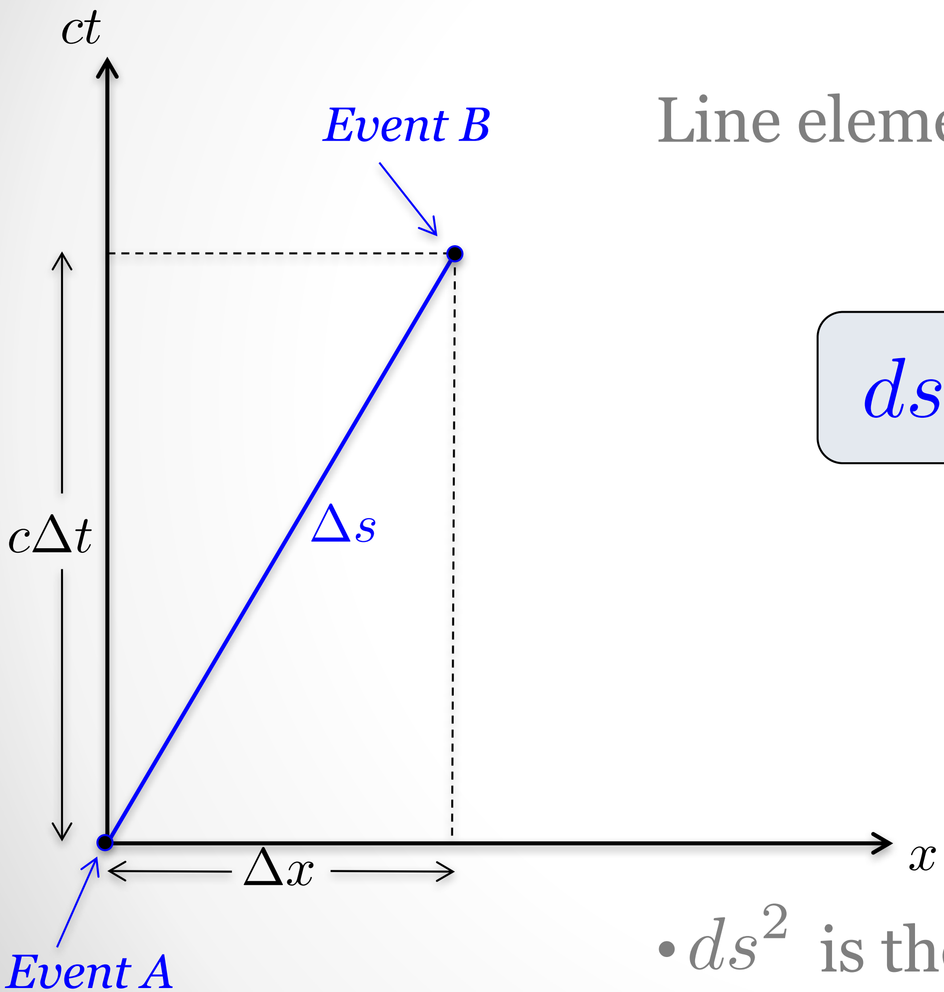


Line element in Minkowski (flat) spacetime

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

- ds^2 is the *line element* measuring *'length'*

4D Minkowski Spacetime



Line element in Minkowski (flat) spacetime

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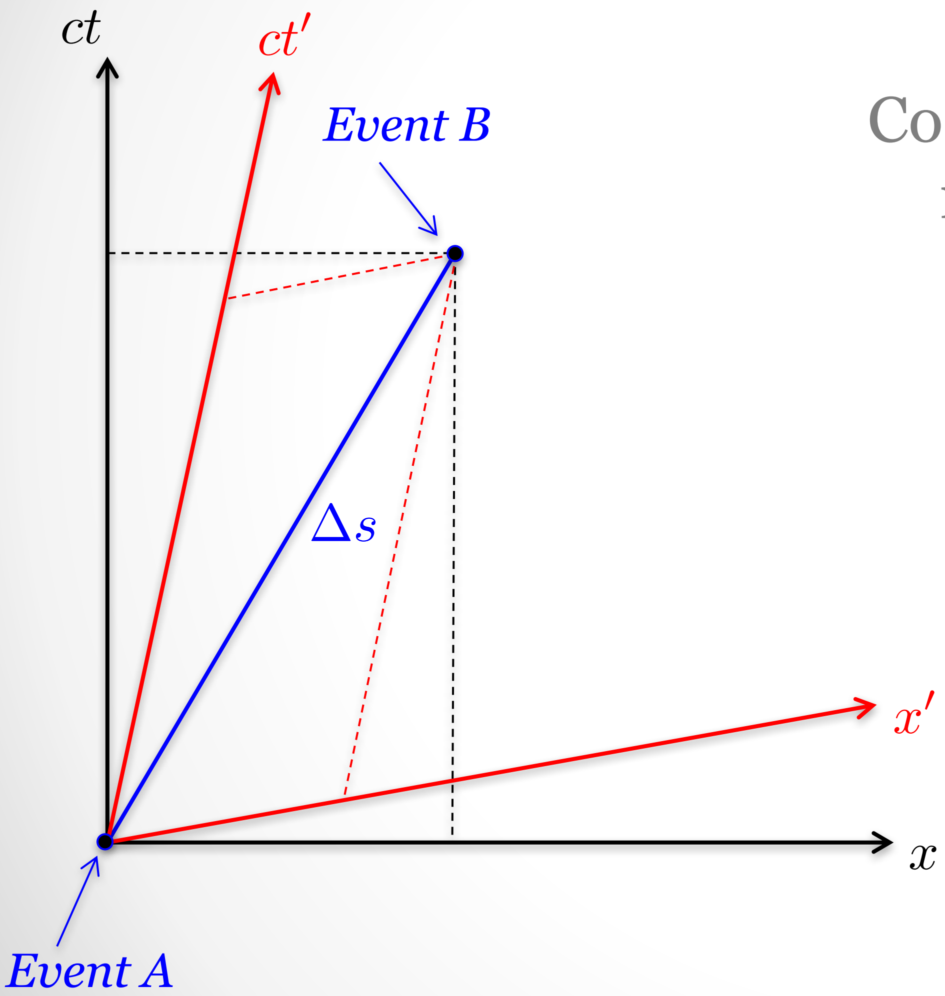
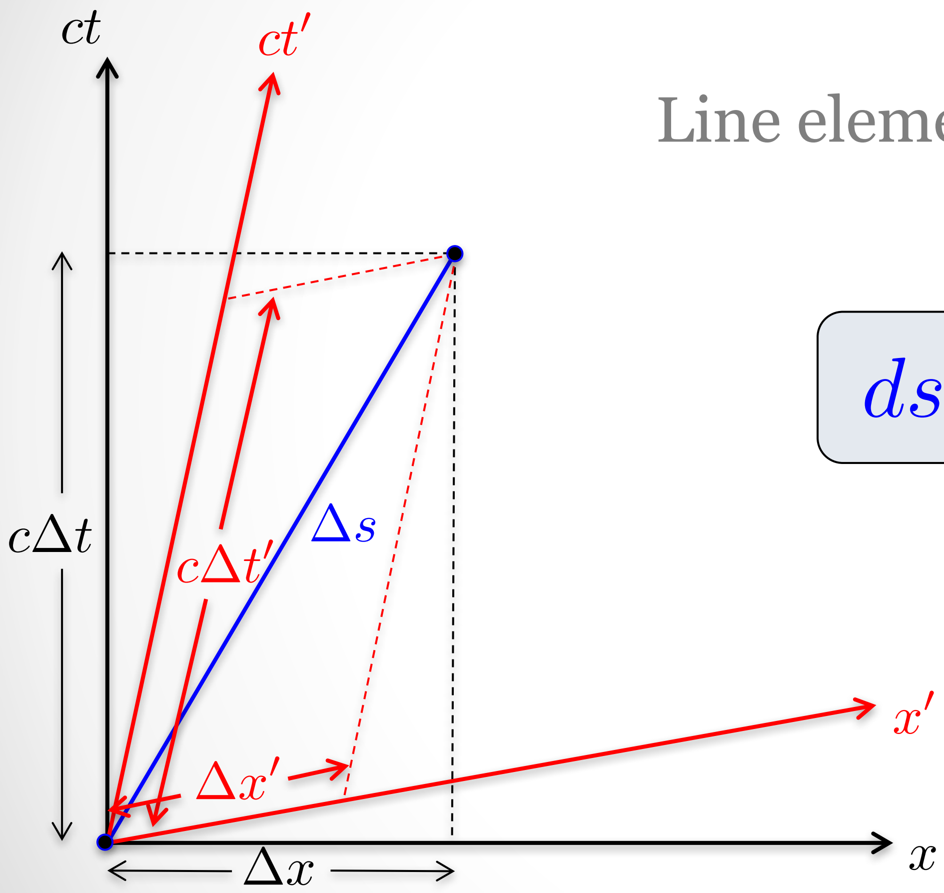
Spacetime diagrams



Consider a *primed* frame moving relative to the *unprimed* frame…

Lorentz Boosts in 4D Minkowski Spacetime



Line element in Minkowski (flat) spacetime

$$ds^2 = -c^2 dt^2 + dx^2$$

$$= -c^2 dt'^2 + dx'^2 = ds'^2$$

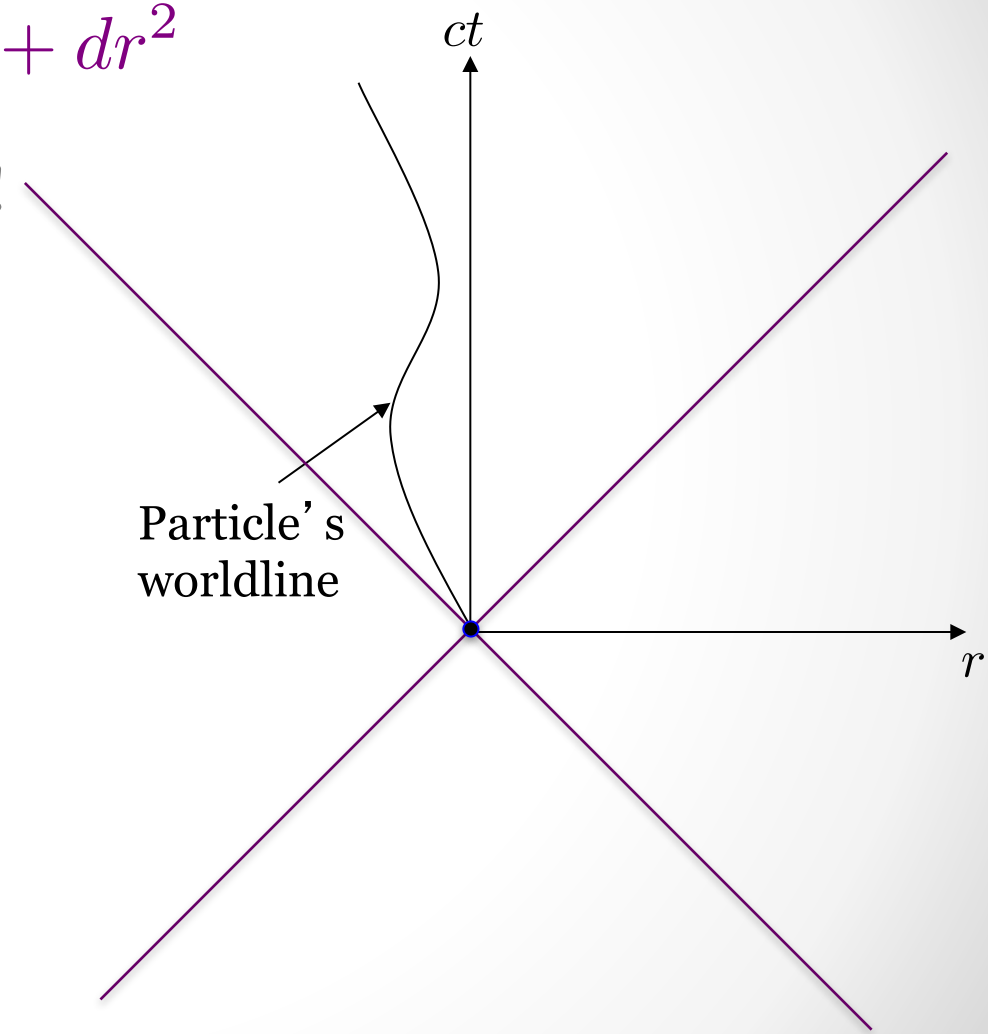
- ds^2 is the *line element* measuring *'length'*
- ds^2 is *invariant* under *'rotations'*

Consider *radial null curves* (θ & ϕ = *const*, $ds^2 = 0$)...

$$ds^2 = 0 = -c^2dt^2 + dr^2$$

->world line of a light beam!

$$c\frac{dt}{dr} = \pm 1$$



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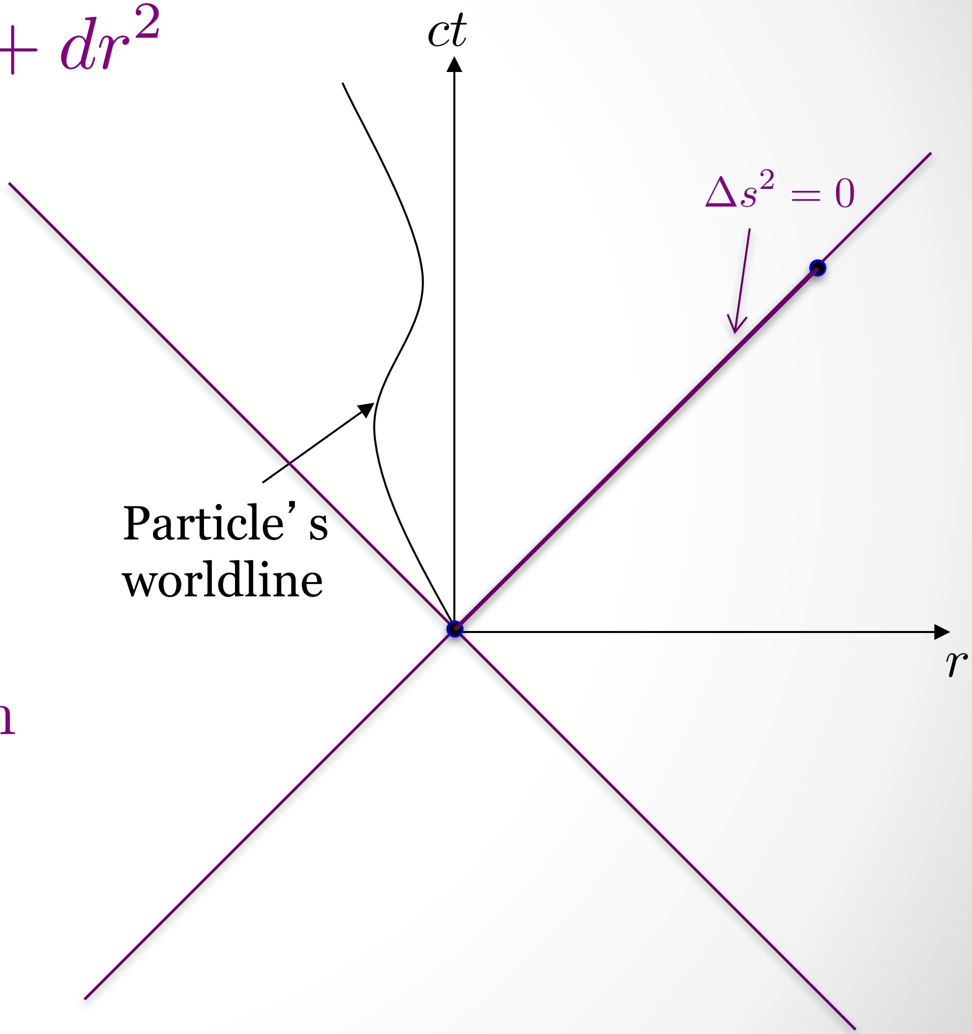
For 2 events..

$ds^2 = 0$: lightlike separation



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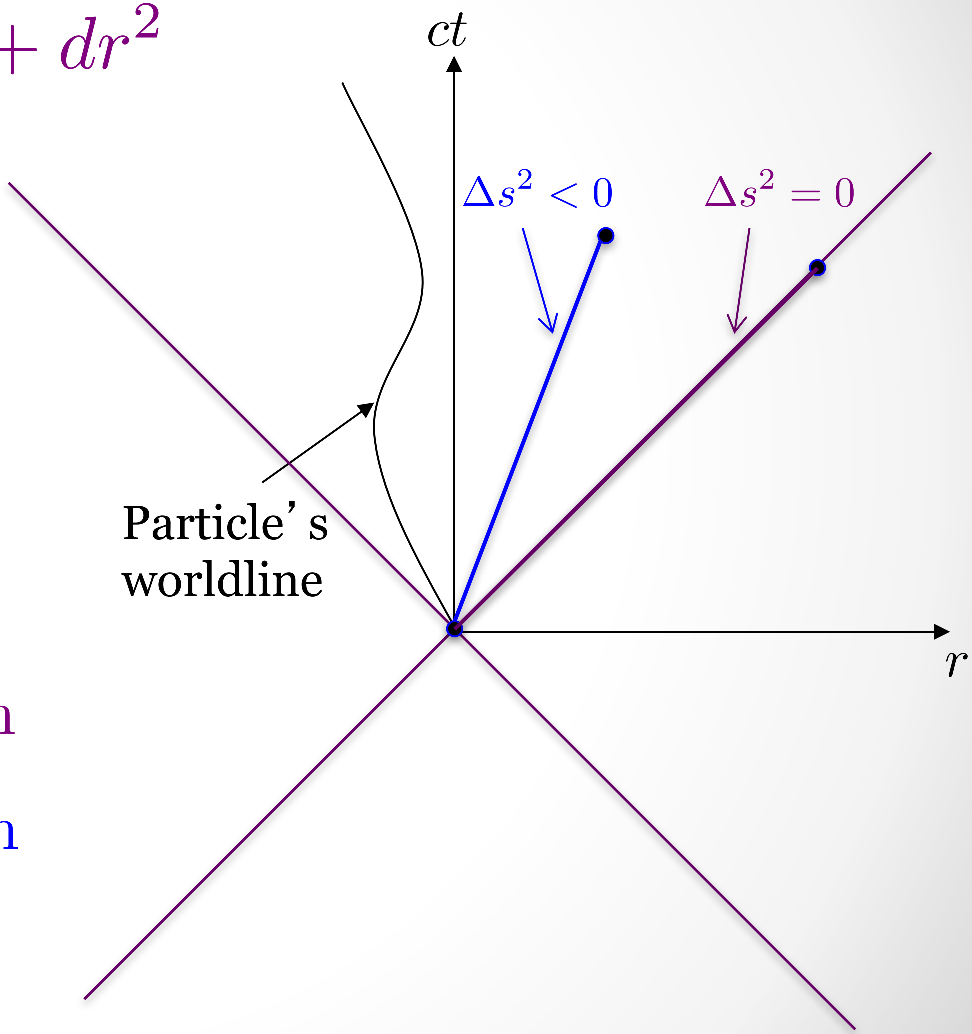


For 2 events..

$ds^2 = 0$: lightlike separation

$ds^2 < 0$: timelike separation

Consider *radial null curves* (θ & ϕ = *const*, $ds^2 = 0$)...

$$ds^2 = 0 = -c^2 dt^2 + dr^2$$

->world line of a light beam!

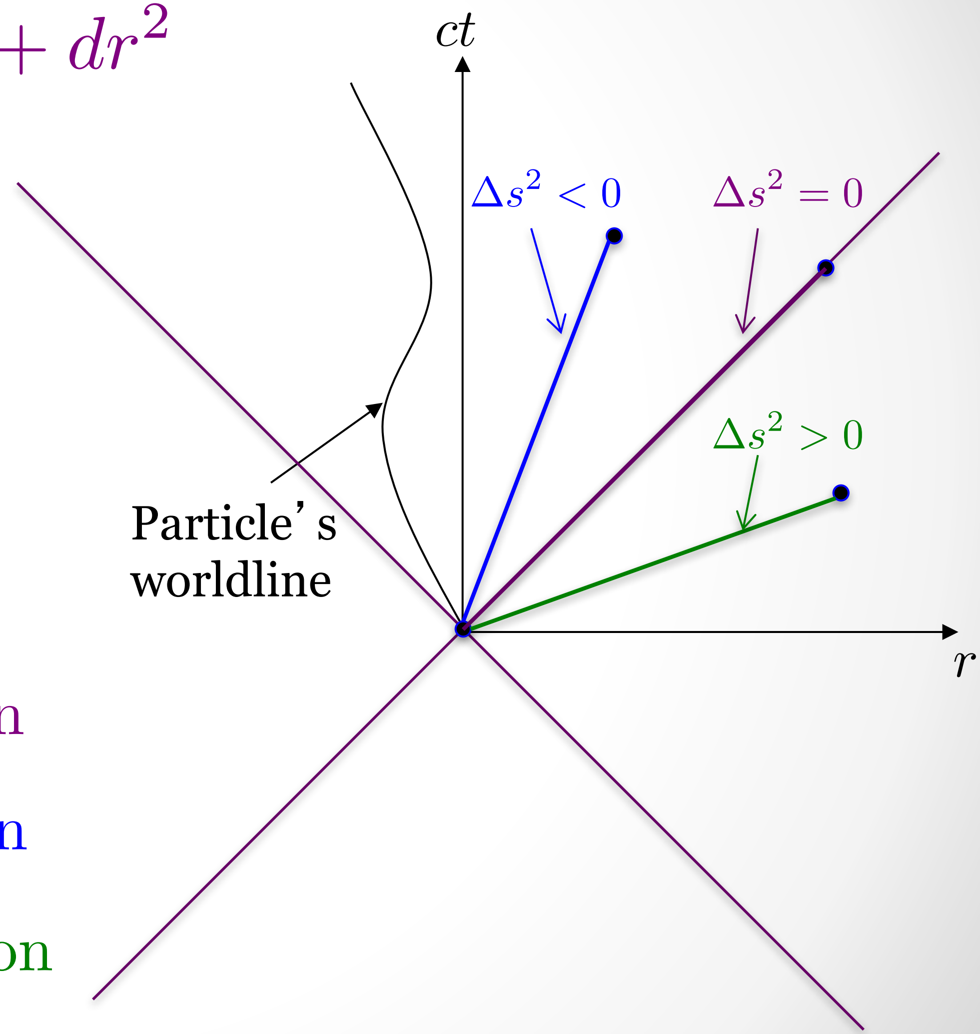
$$c\frac{dt}{dr} = \pm 1$$

For 2 events..

$ds^2 = 0$: lightlike separation

$ds^2 < 0$: timelike separation

$ds^2 > 0$: spacelike separation



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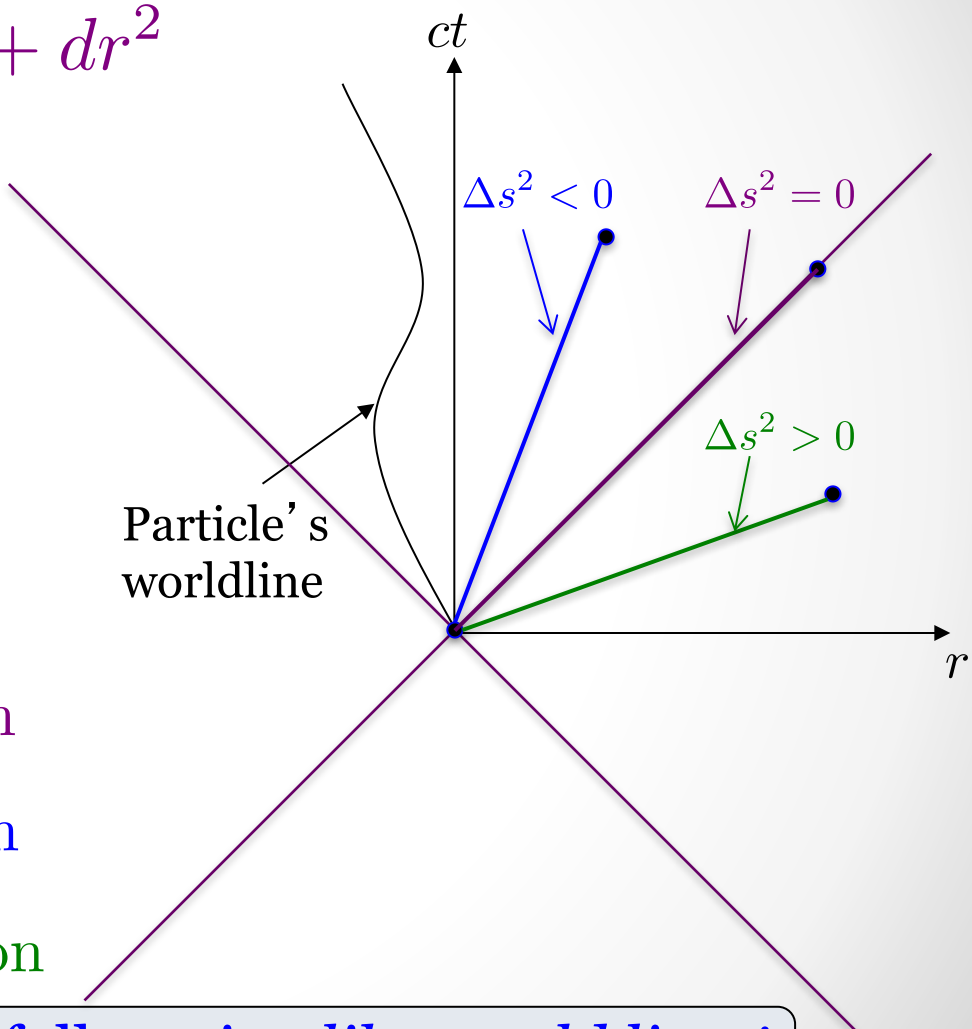


For 2 events..

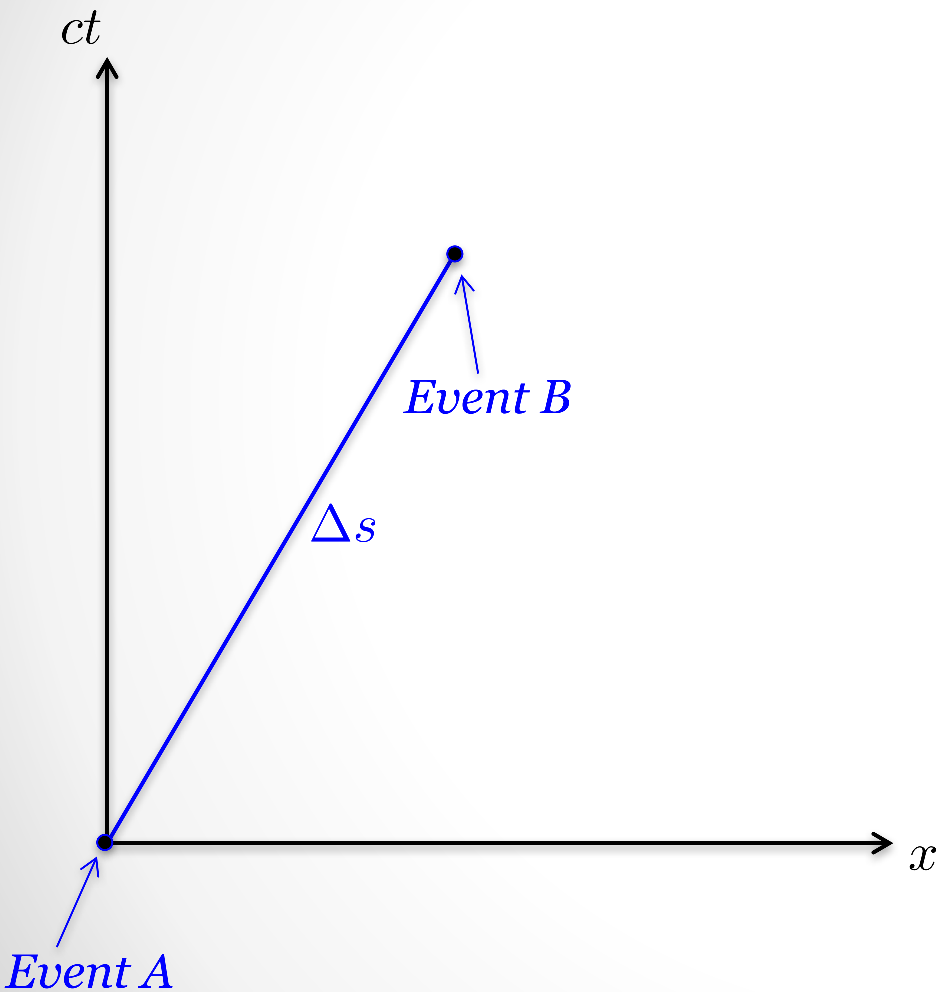
$ds^2 = 0$: lightlike separation

$ds^2 < 0$: timelike separation

$ds^2 > 0$: spacelike separation

Massive particles follow *timelike world lines*!

Lorentz Boosts in 4D Minkowski Spacetime

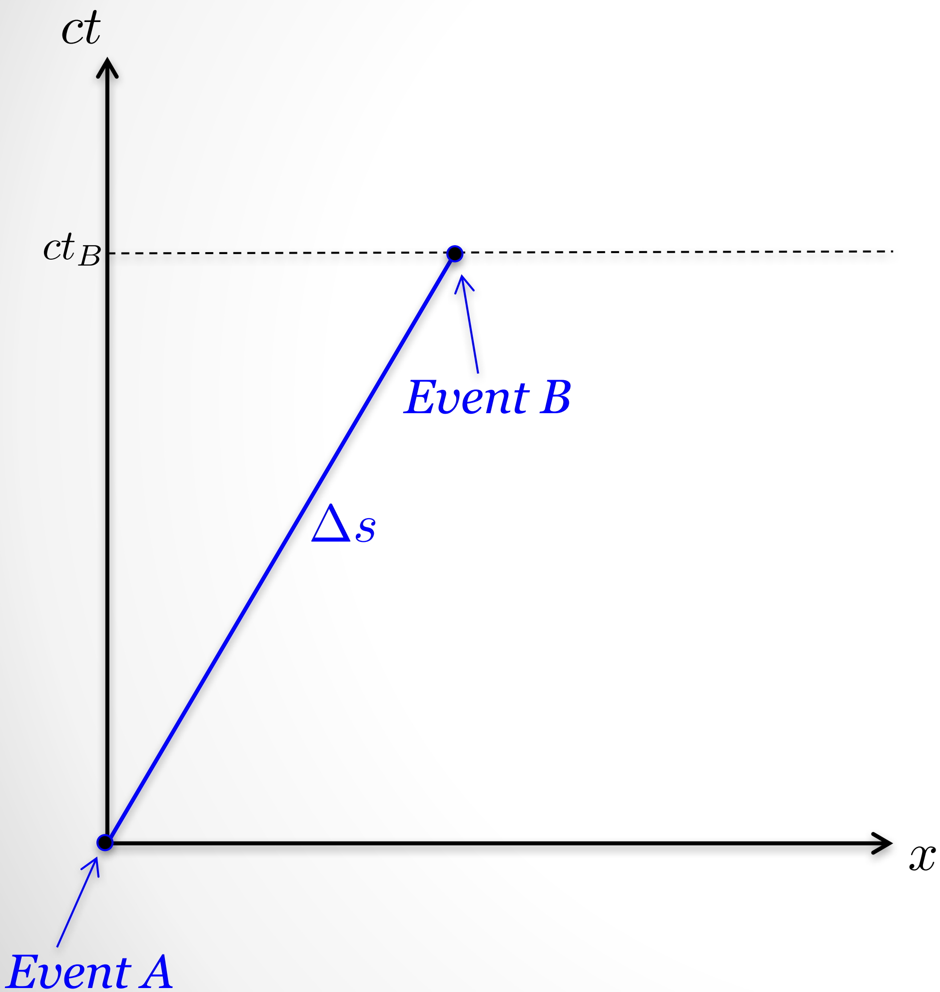


Consider 2 *timelike* separated events…

Notice:

-
-
-

Lorentz Boosts in 4D Minkowski Spacetime



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Notice:

- $t_B > t_A$
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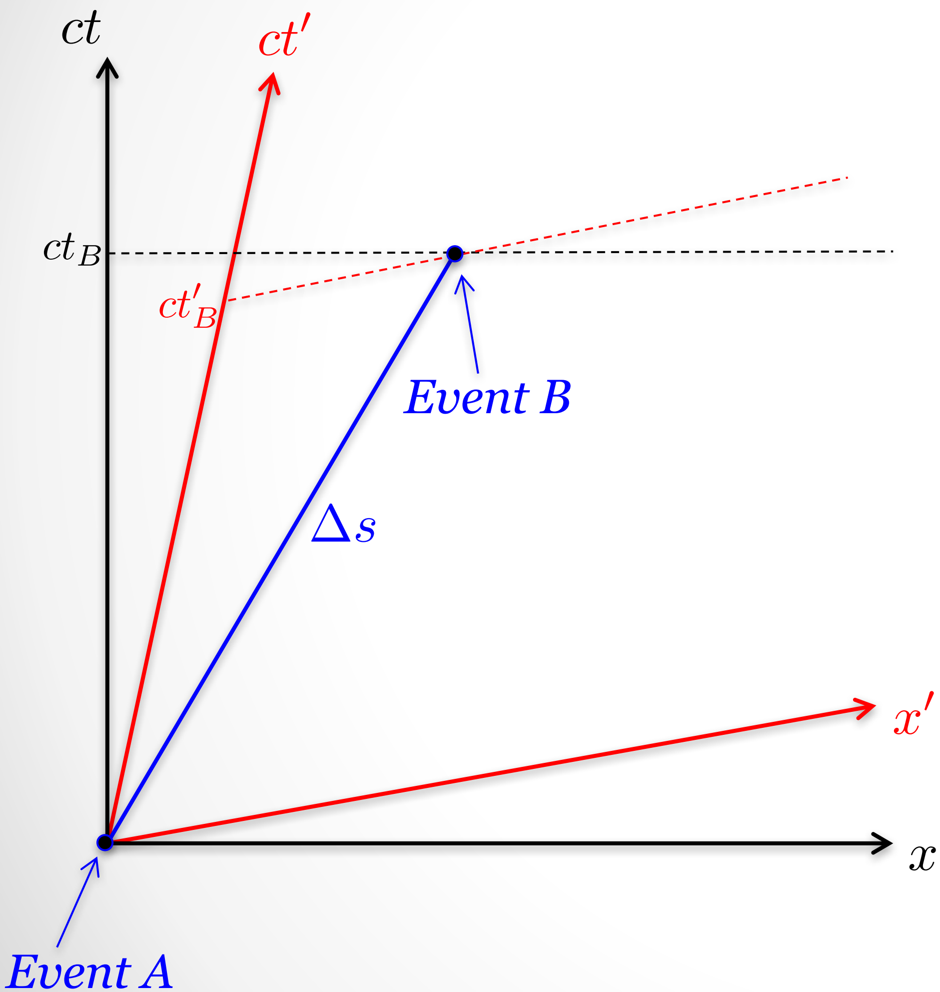
Lorentz Boosts in 4D Minkowski Spacetime



Consider 2 *timelike* separated events…

Notice:

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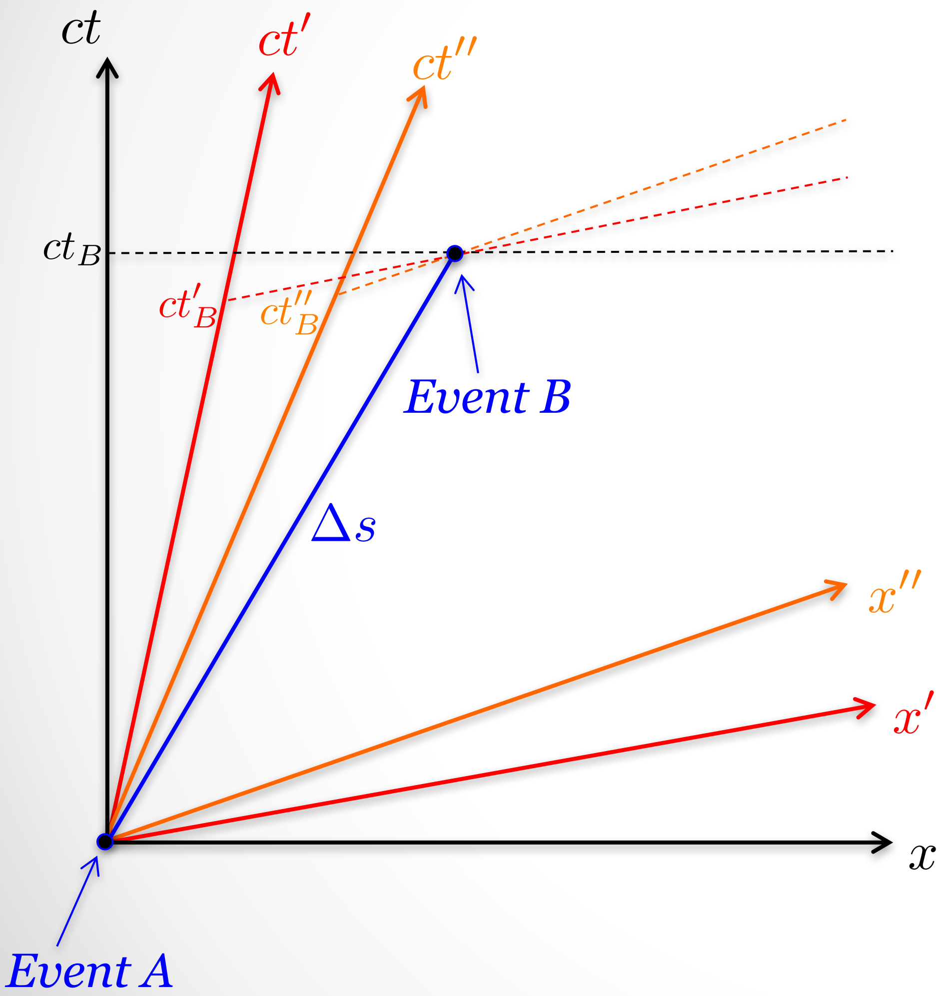
Lorentz Boosts in 4D Minkowski Spacetime



Consider 2 *timelike* separated events...

Notice:

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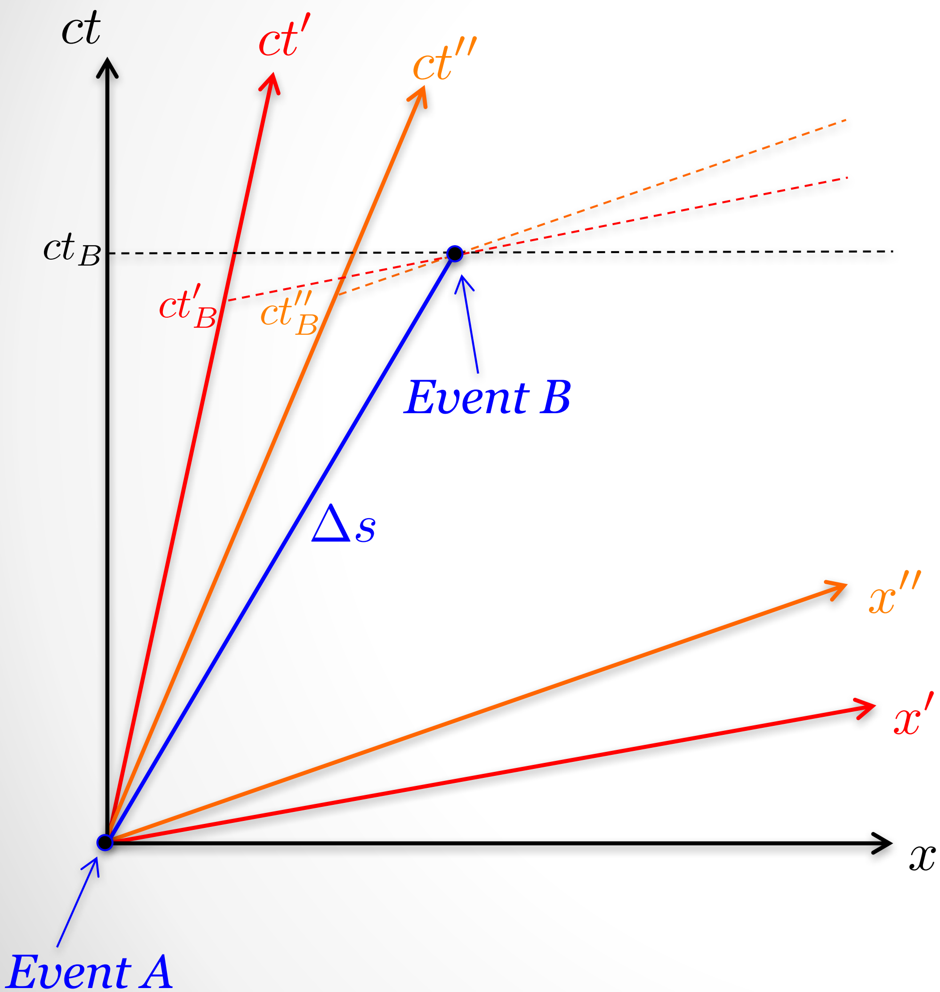
Lorentz Boosts in 4D Minkowski Spacetime



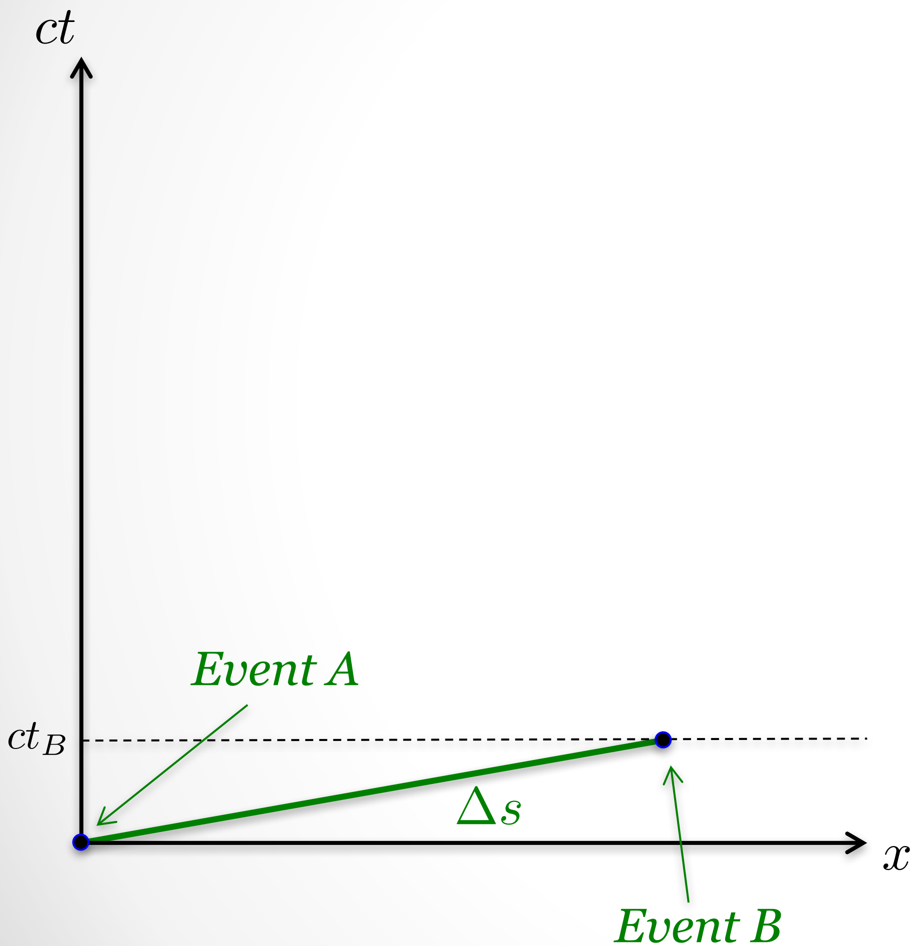
Consider 2 *timelike* separated events…

Notice:

- $t_B > t_A$
- $t'_B > t'_A$
- $t''_B > t''_A$

Time ordering is preserved between *timelike* separated events!

Lorentz Boosts in 4D Minkowski Spacetime

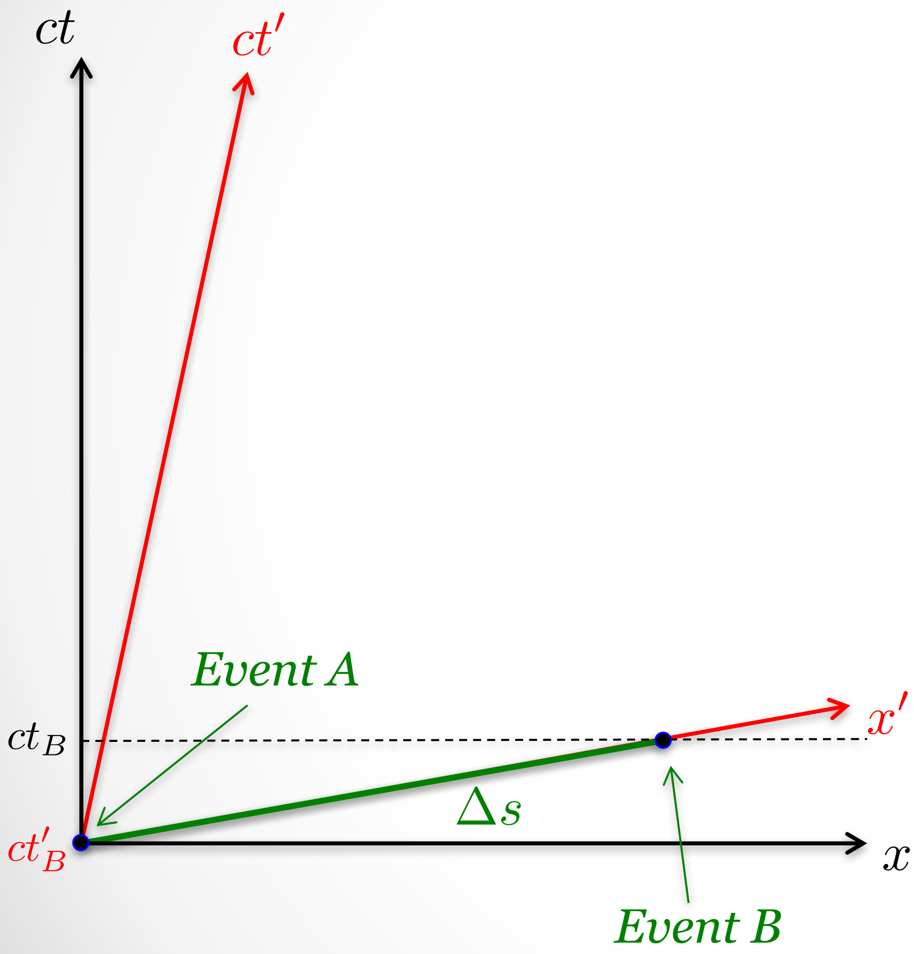


Consider 2 *spacelike* separated events…

Notice:

- $t_B > t_A$
-
-

Lorentz Boosts in 4D Minkowski Spacetime



Consider 2 *spacelike* separated events…

Notice:

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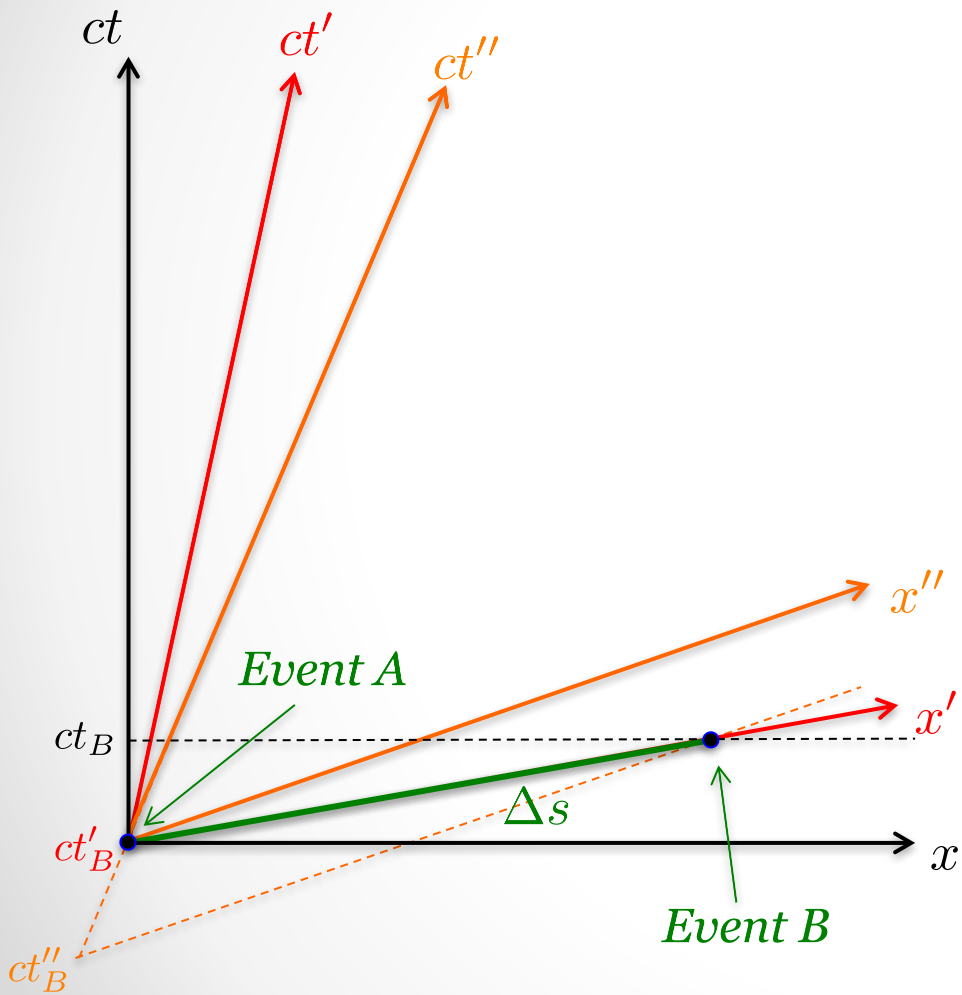
Lorentz Boosts in 4D Minkowski Spacetime



Consider 2 *spacelike* separated events…

Notice:

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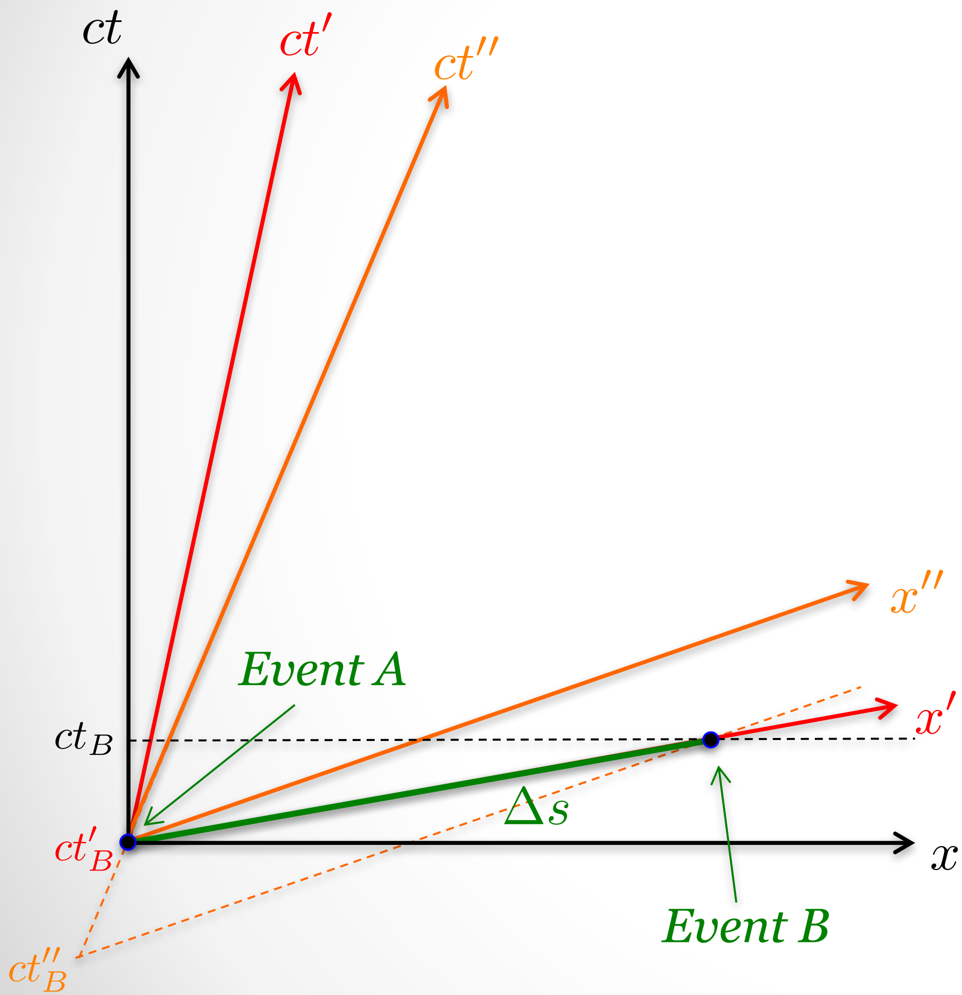
Lorentz Boosts in 4D Minkowski Spacetime



Consider 2 *spacelike* separated events…

Notice:

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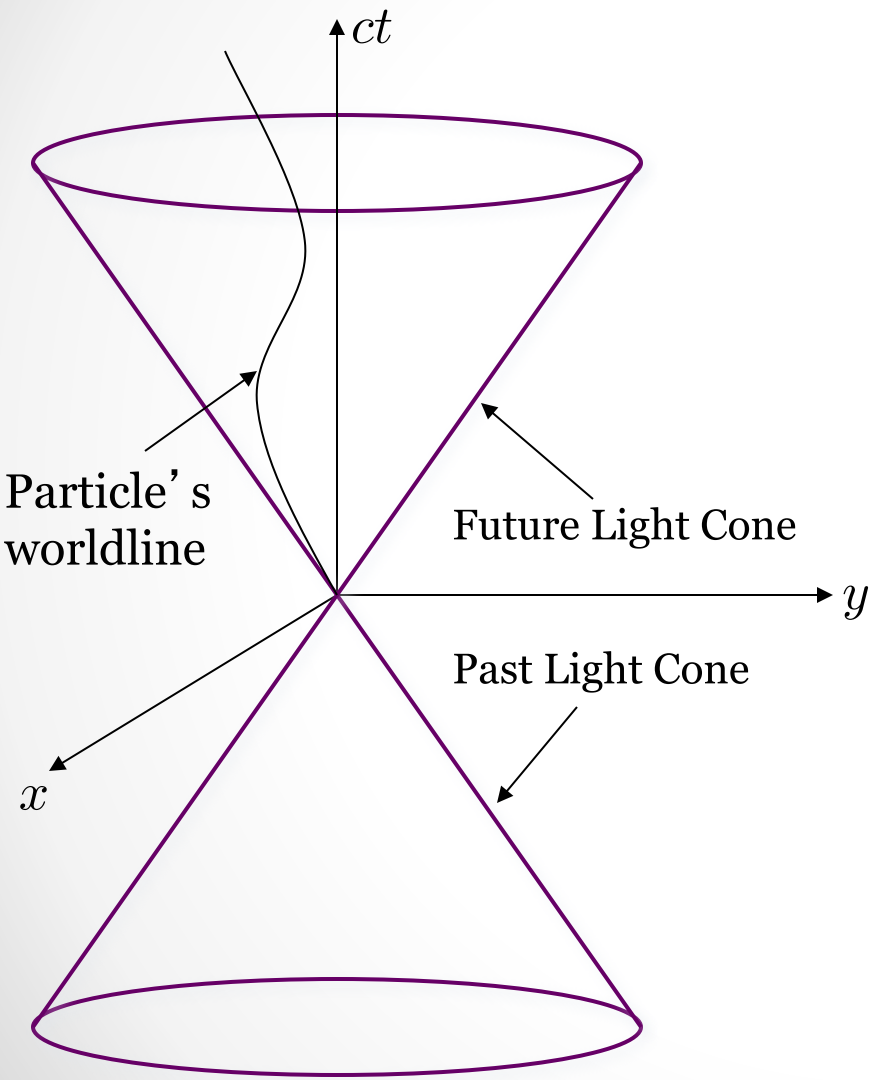
Time ordering is NOT preserved between *spacelike* separated events!

Spacetime diagram & light cones



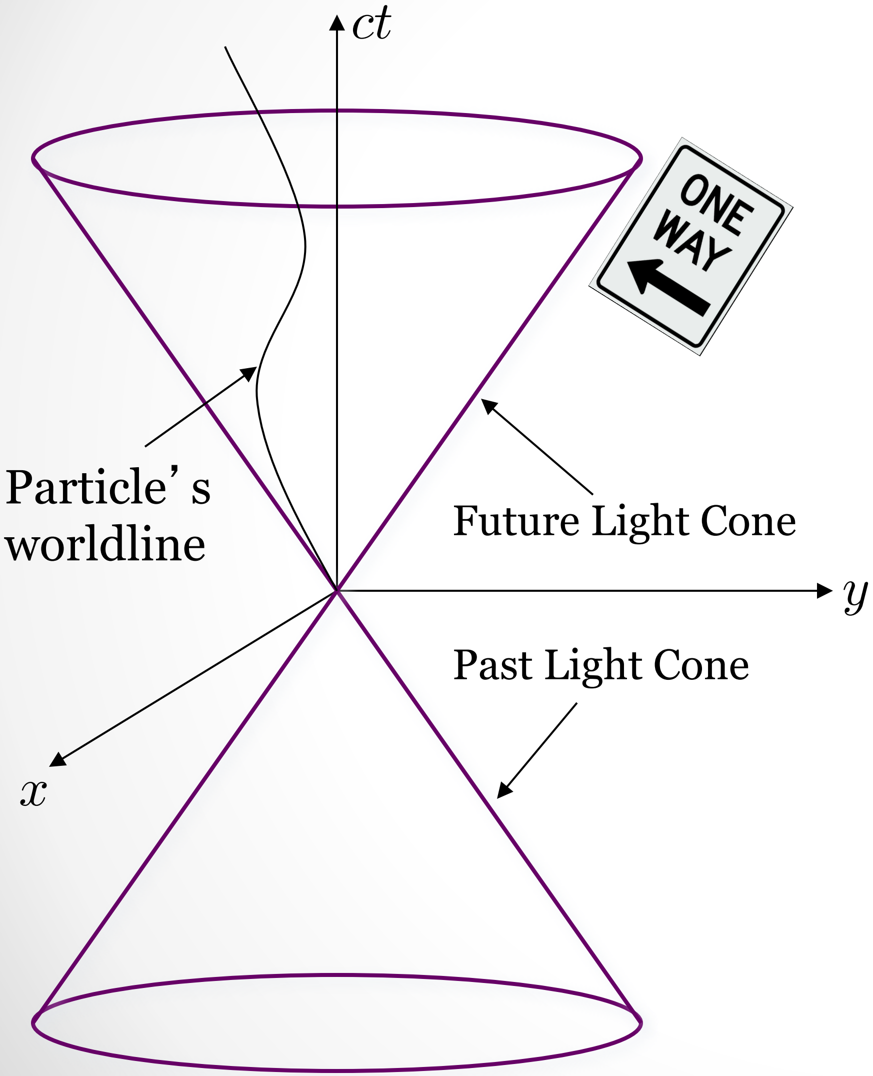
Minkowski line element in spherical coordinates…

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$

Notice:

- For *constant* time slice, spherical wave front

Spacetime diagram & light cones



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- Light cone is a *one-way surface*

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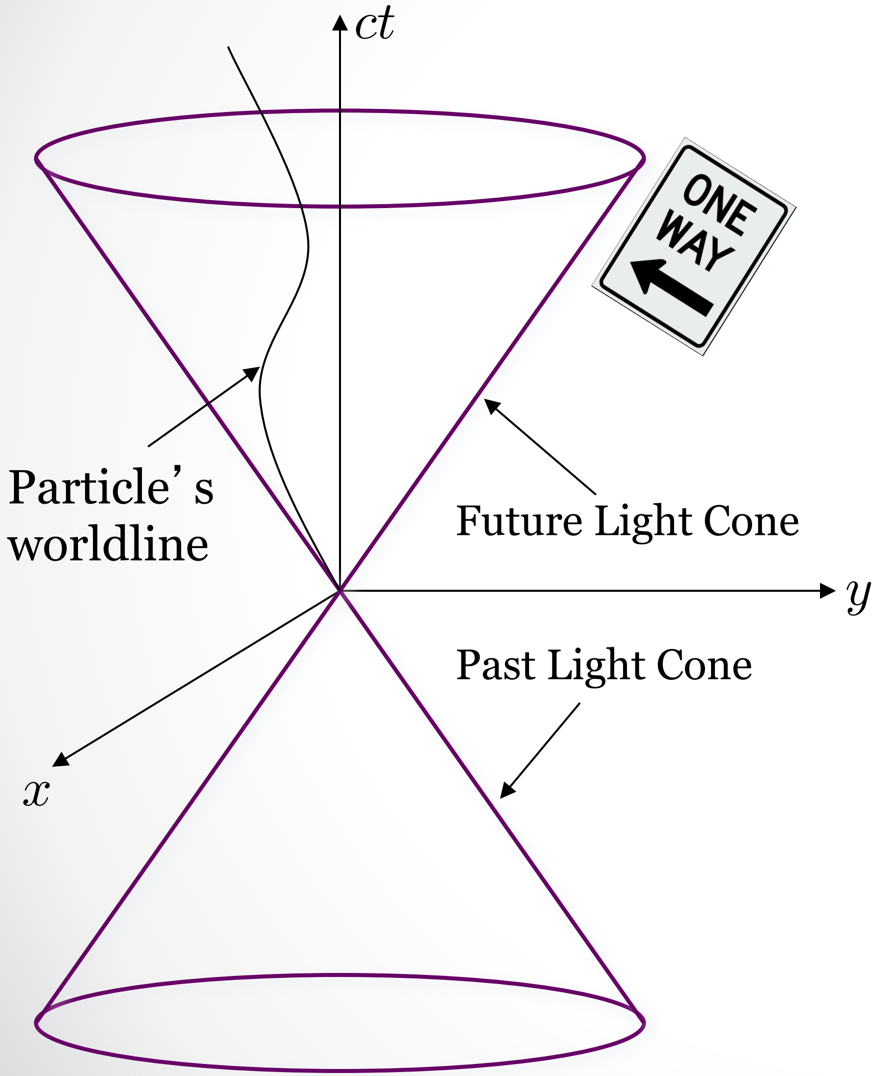


Minkowski line element in spherical coordinates...

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$$

Notice:

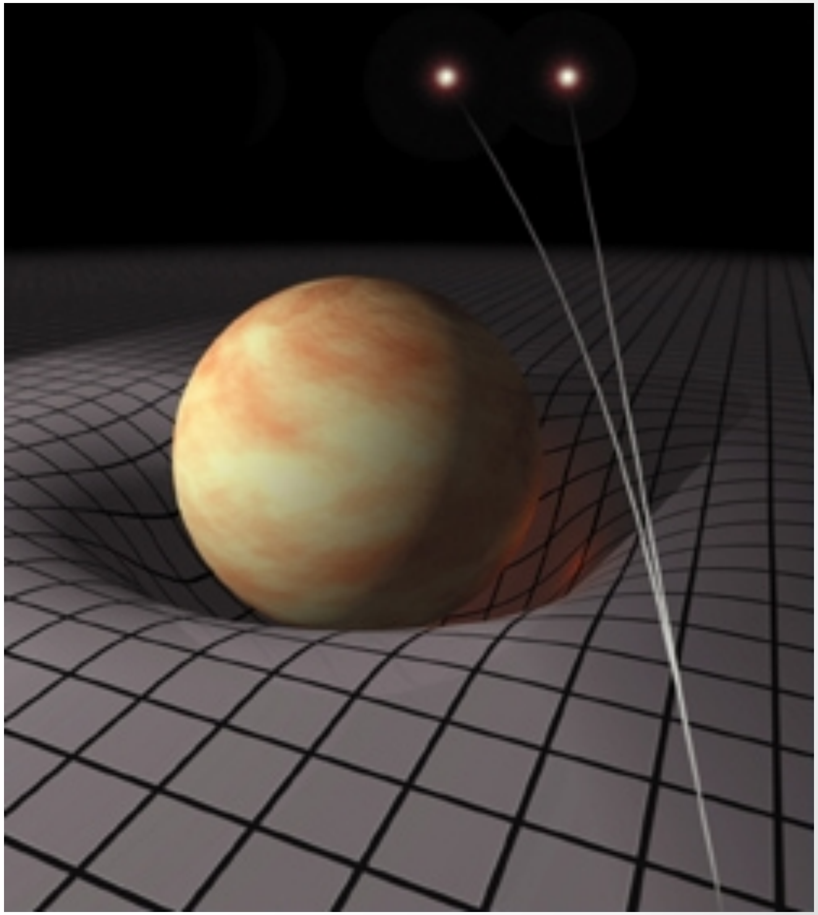
- For *constant* time slice, spherical wave front
- **Light cone is a *one-way surface***
- *Global* arrangement of light cones determine *causal structure* of spacetime

Einstein's theory of general relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $G_{\mu\nu}$ describes the *curvature of spacetime*
- $T_{\mu\nu}$ describes the *matter & energy in spacetime*

Matter tells *space* how to curve
Space tells *matter* how to move



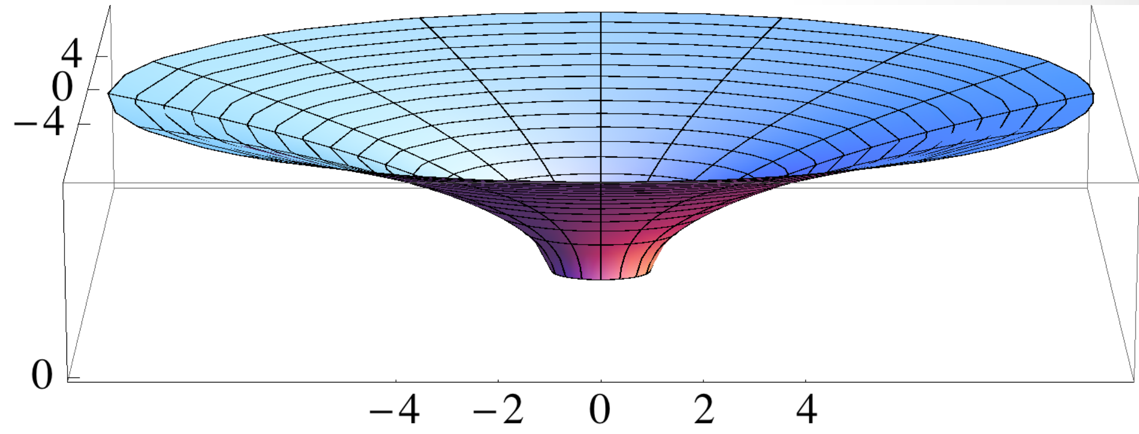
Sean M. Carrol, *Spacetime and Geometry: An Introduction to Einstein's General Relativity* (Addison Wesley, 2004)

The spherically-symmetric, time-independent, vacuum solution to GR is…

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Let:

$$f(r) \equiv 1 - \frac{2GM}{c^2 r}$$



Notice:

$$\lim_{r \to 2GM/c^2} f(r) \to 0 \quad : \textit{singularity}$$

$$\lim_{r \to 0} f(r) \to \infty \quad : \textit{singularity}$$

Coordinate vs physical singularities…

Calculate an *invariant* scalar quantity:

$$I \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12 \left(\frac{2GM}{c^2 r^3} \right)^2$$

Notice:

$$\lim_{r \to 2GM/c^2} I \to \frac{12}{R_s^4}$$ *: well behaved (coordinate singularity)*

$$\lim_{r \to 0} I \to \infty$$ *: divergent (physical singularity)*

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Notice:

$$\lim_{r \to 2GM/c^2} I \to \frac{12}{R_s^4}$$ *: well behaved (coordinate singularity)*

$$\lim_{r \to 0} I \to \infty$$ *: divergent (physical singularity)*

The Schwarzschild coordinate system is *pathological* at $r = 2GM/c^2$ & MUST be abandoned there!

Black holes…

- For fixed radius *R*…

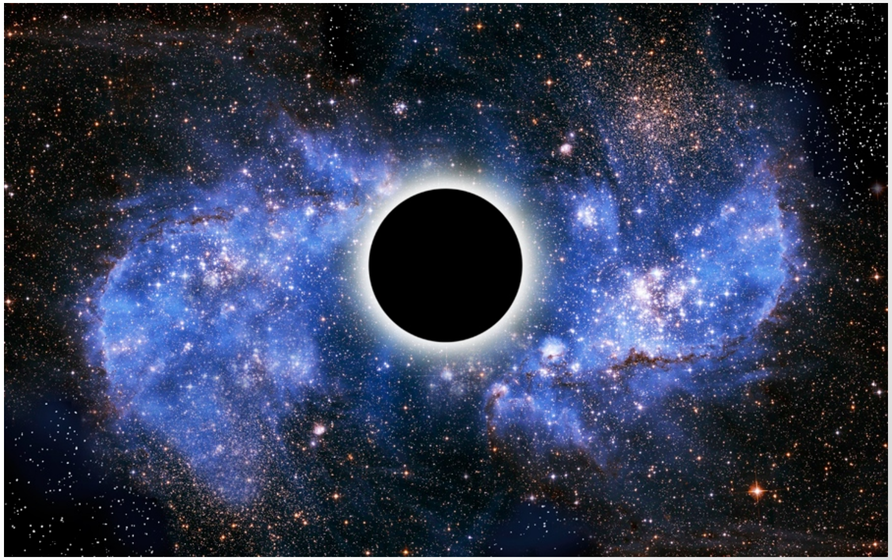
$$p(r=0) \to \infty \quad \text{when} \quad M_{max} = \frac{4c^2}{9G}R$$

- For M_{star}~3-4 M_{sun}, star collapses to a black hole!

- *Black holes* have an *event horizon (a.k.a. Schwarzschild radius):*

$$R_s = \frac{2GM}{c^2}$$

- *Black holes don't suck!*
(External geometry of a black hole is the *same* as that of a star or planet)



http://www.universetoday.com/33454/how-do-black-holes-form/

Radial plunge of an *experimental* physicist..

$$\tau = \tau_* - \frac{2}{3}\frac{R_s}{c}\left(\frac{r}{R_s}\right)^{3/2}$$ - *Proper time*

$$t = t_* + \frac{R_s}{c}\left[-\frac{2}{3}\left(\frac{r}{R_s}\right)^{3/2} - 2\left(\frac{r}{R_s}\right)^{1/2} + \ln\left[\frac{\sqrt{r/R_s}+1}{\sqrt{r/R_s}-1}\right]\right]$$ - *Coordinate time*

Notice:

$$\lim_{r\to R_s} \tau \to \text{finite}$$

$$\lim_{r\to R_s} t \to \infty$$

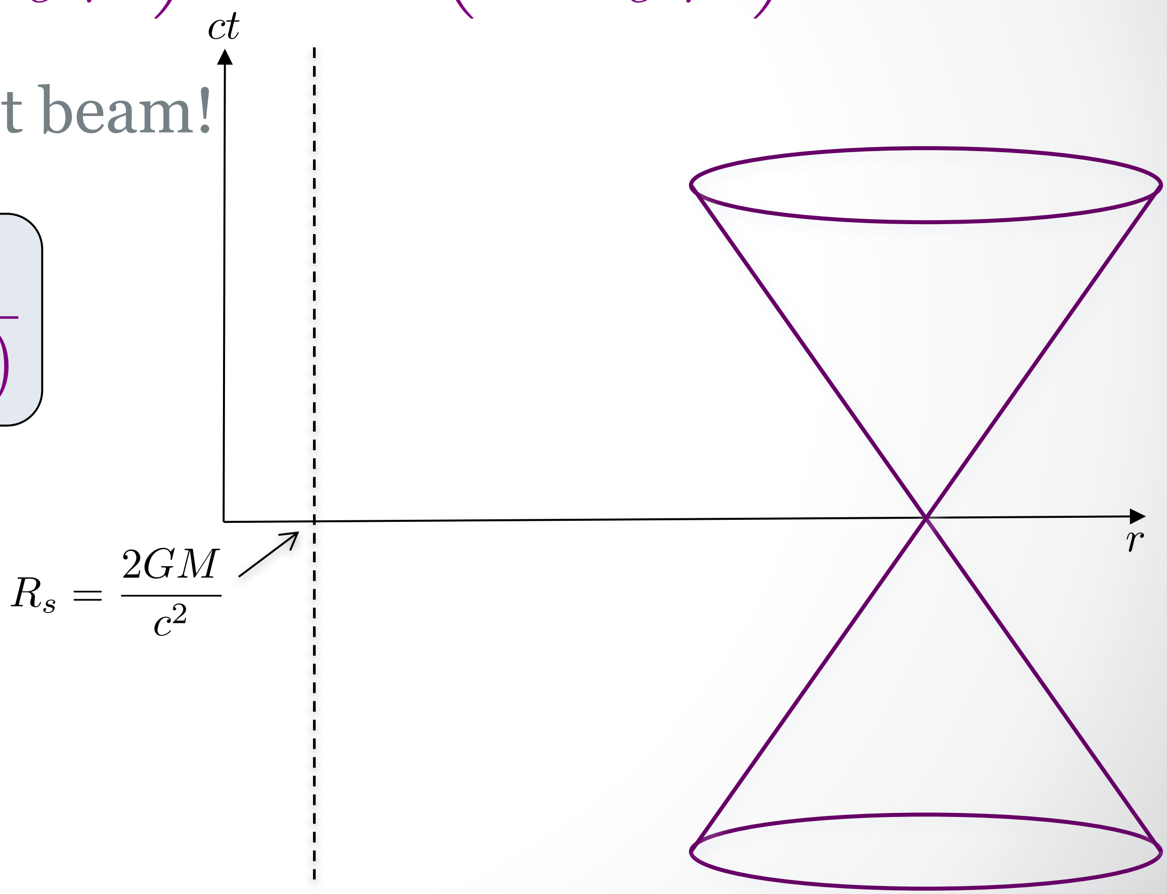
It takes a *finite* amount *proper time*, τ, to reach $r = R_s$ but an *infinite* amount of *coordinate time*, t !

Consider *radial null curves* (θ & ϕ = *const*, $ds^2 = 0$)...

$$ds^2 = 0 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$$

-> world line of a light beam!

$$c\frac{dt}{dr} = \pm \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)}$$



Notice:

$$\lim_{r \to R_s} c\frac{dt}{dr} \to \pm\infty$$

Consider *radial null curves* (θ & ϕ = *const*, $ds^2 = 0$)...

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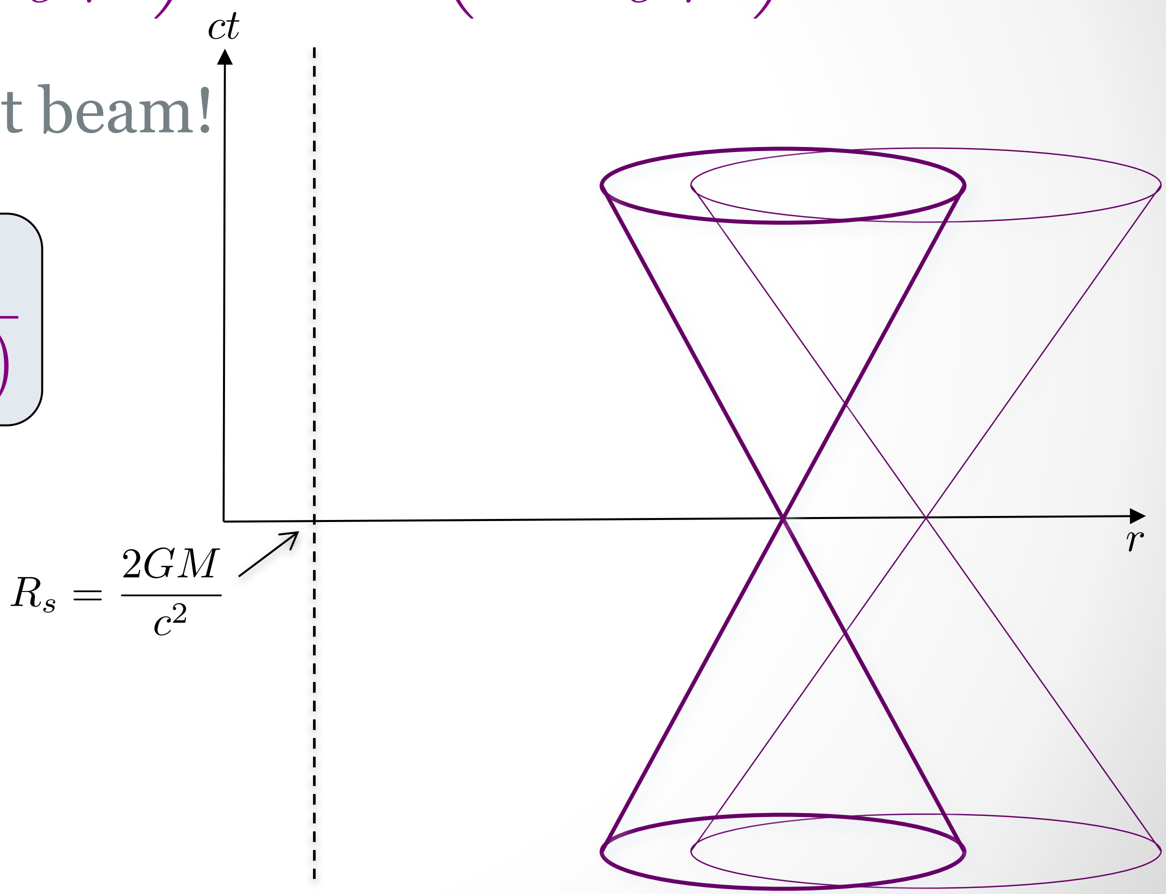

Notice:

$$\lim_{r \to R_s} c\frac{dt}{dr} \to \pm\infty$$

Consider *radial null curves* (θ & ϕ = *const*, $ds^2 = 0$)...

$$ds^2 = 0 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$$

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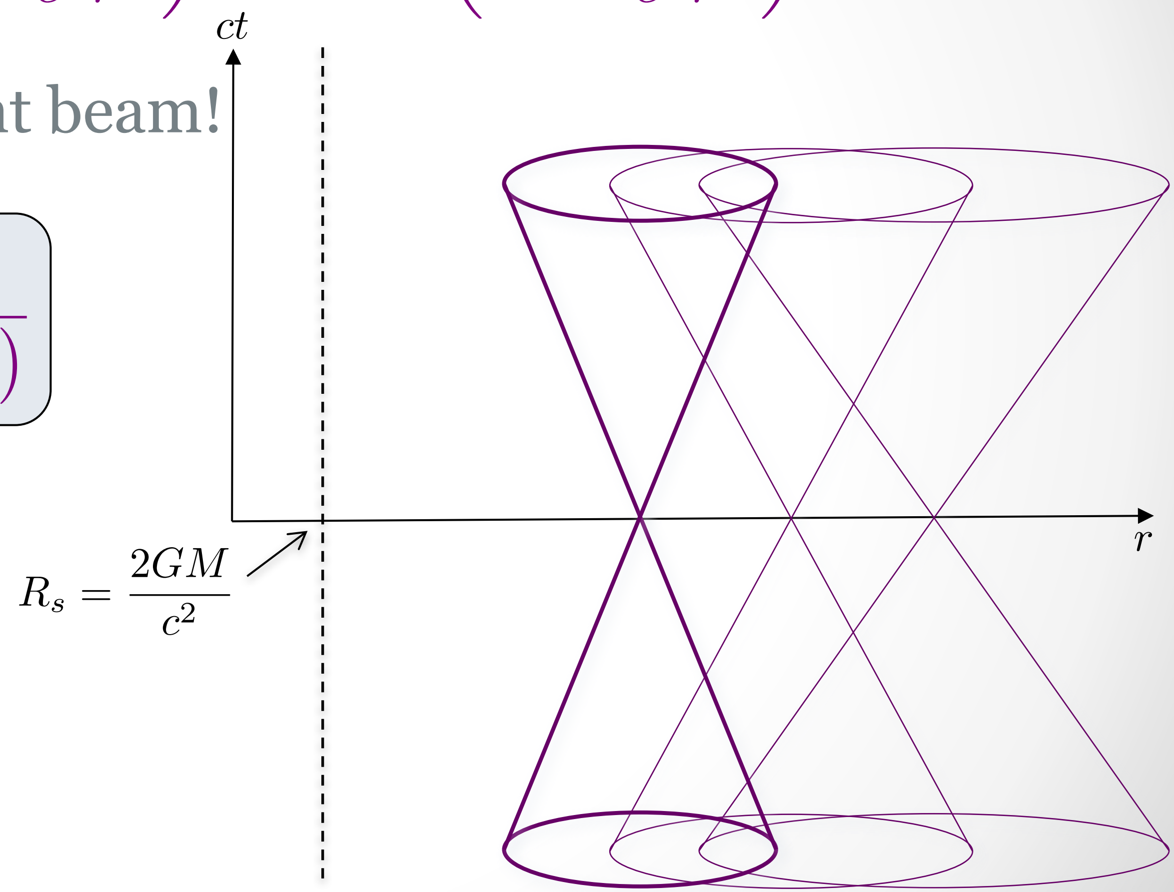

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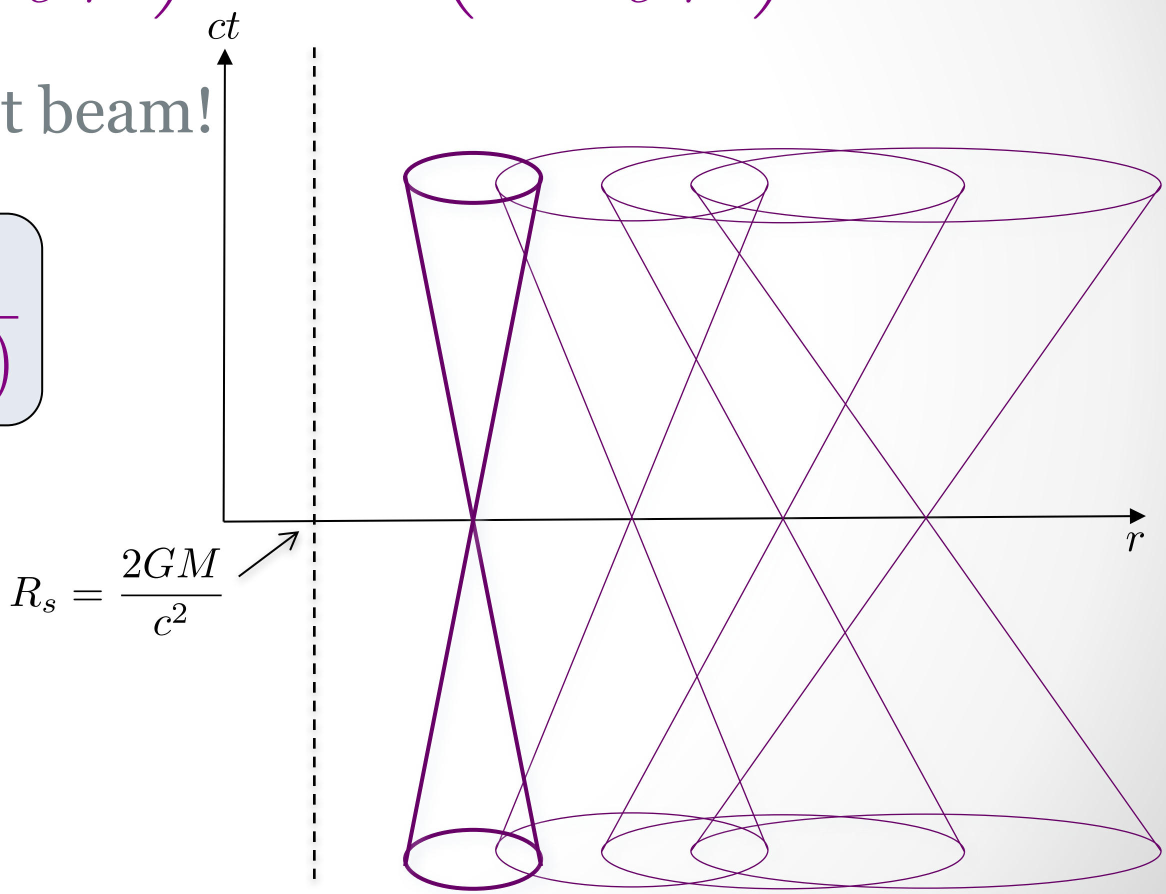

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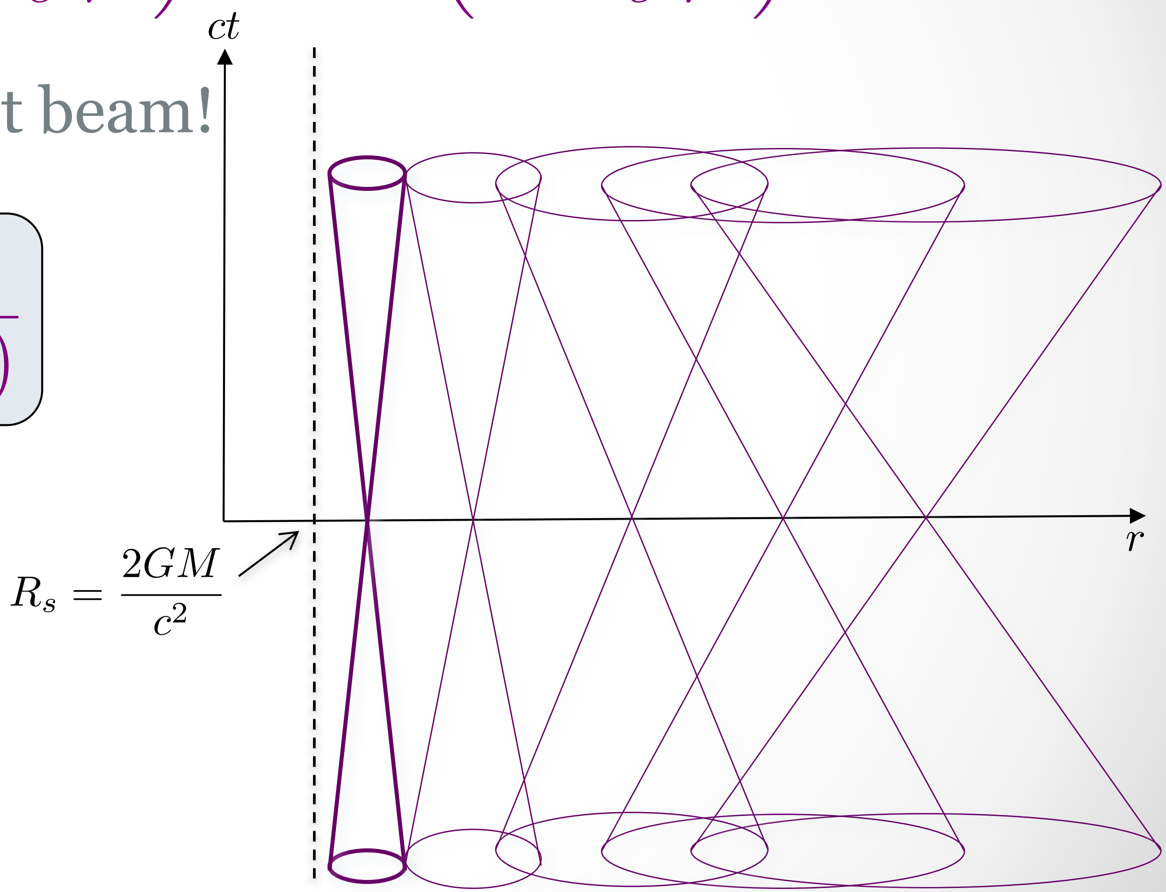

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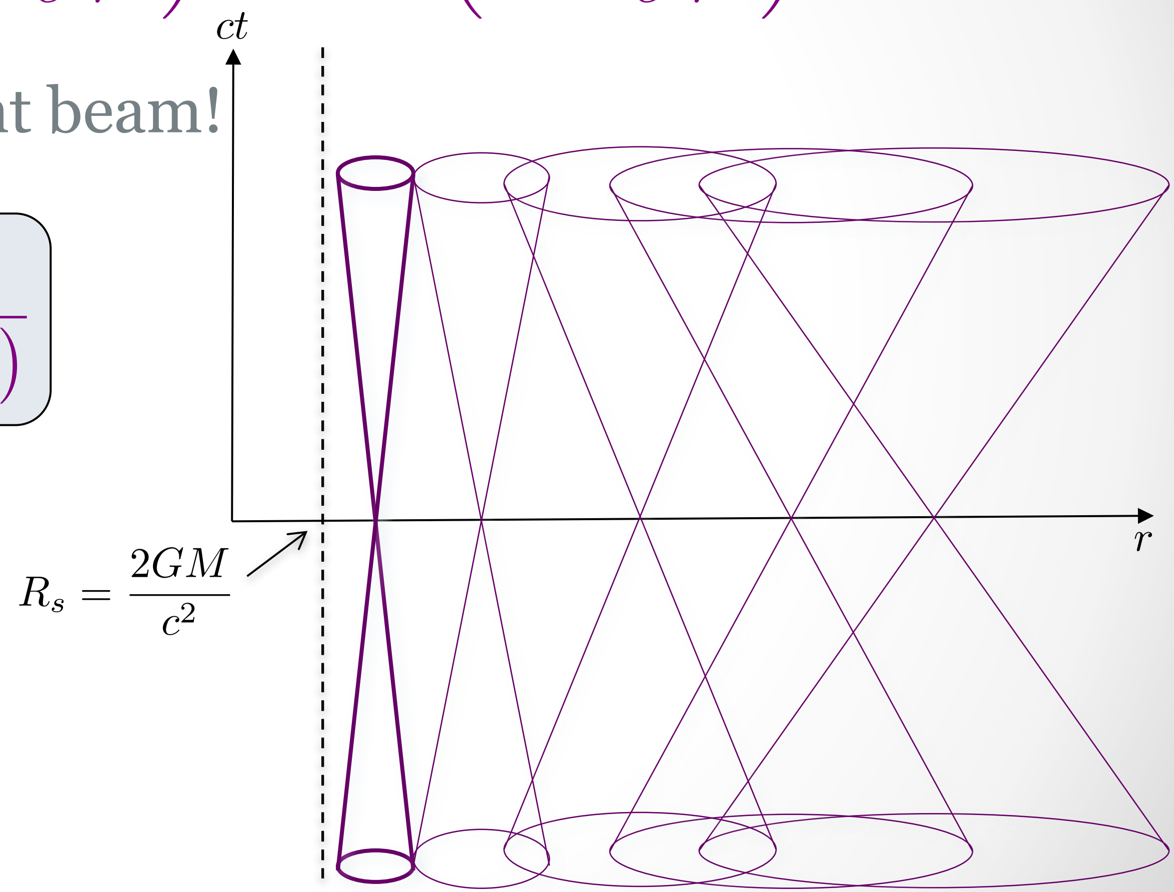


Light cones "close up" at $r = 2GM/c^2$!

In Eddington-Finkelstein Coordinates…

$$ds^2 = 0 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 + 2dvdr$$

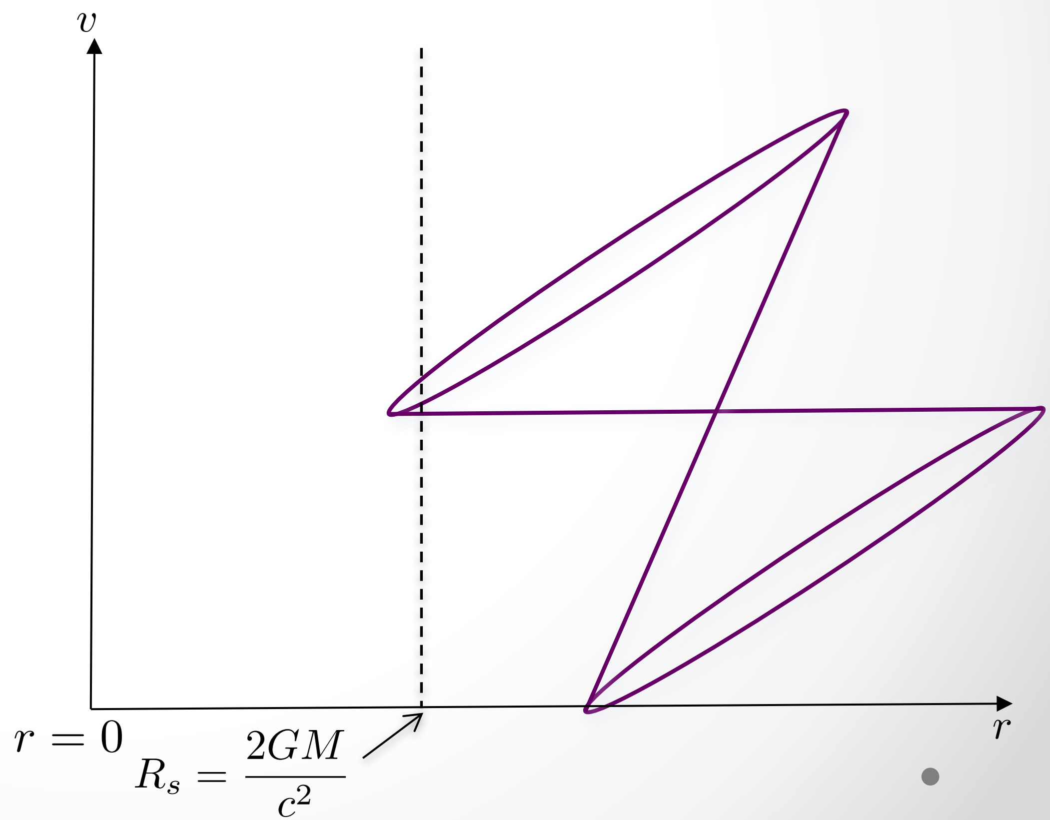
The *slope* of the light cone structure of spacetime is…

$$\frac{dv}{dr} = 0 \quad , \quad \text{(ingoing)}$$

$$\frac{dv}{dr} = \frac{2}{(1 - \frac{2GM}{c^2 r})}$$

Notice:

$$\frac{dv}{dr}|_{r>R_s} = (0, +)$$



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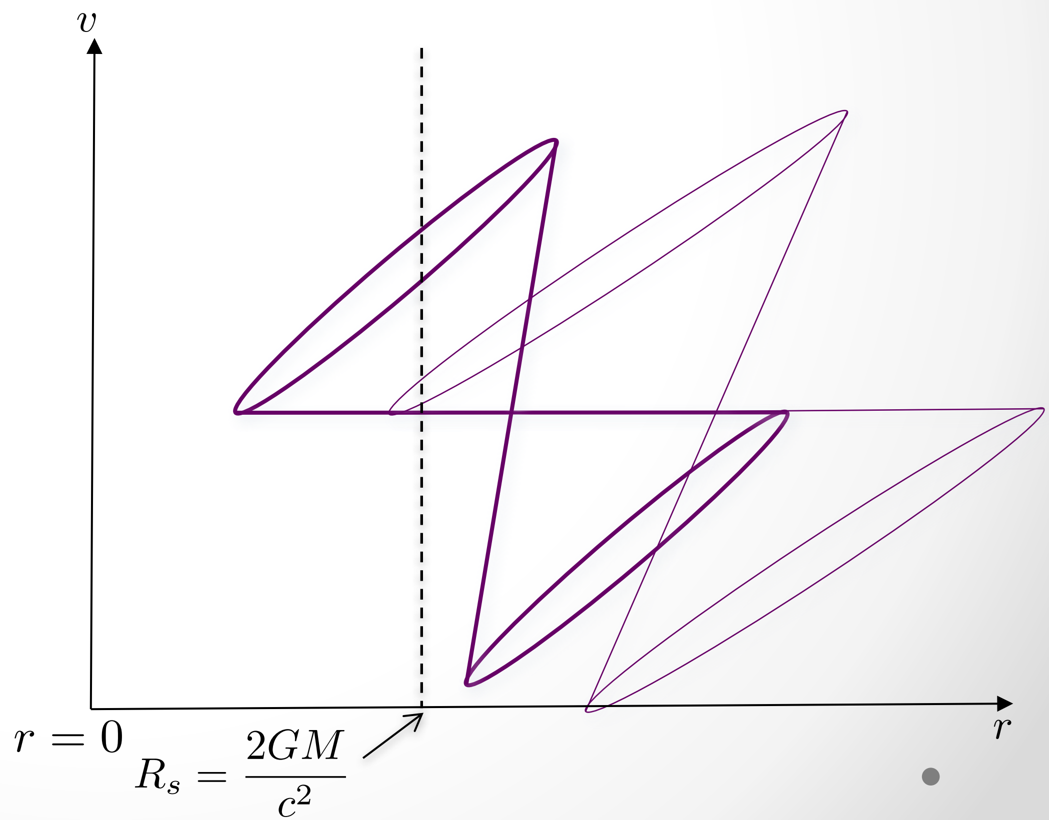

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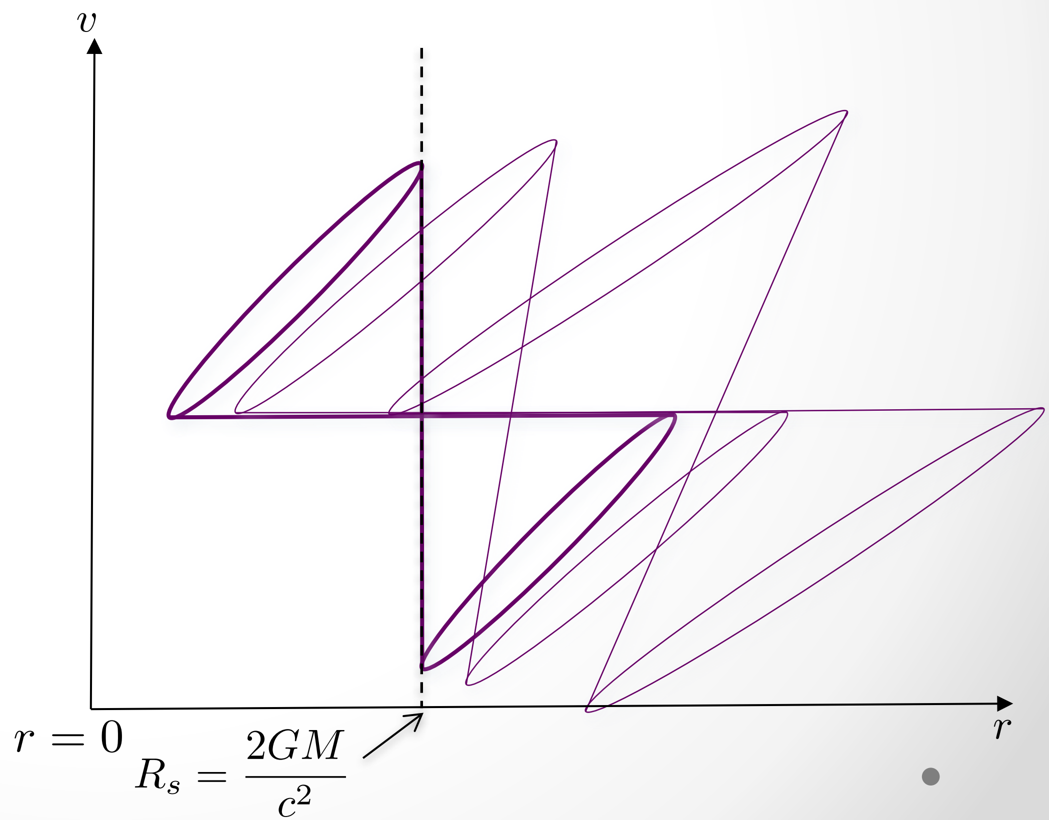

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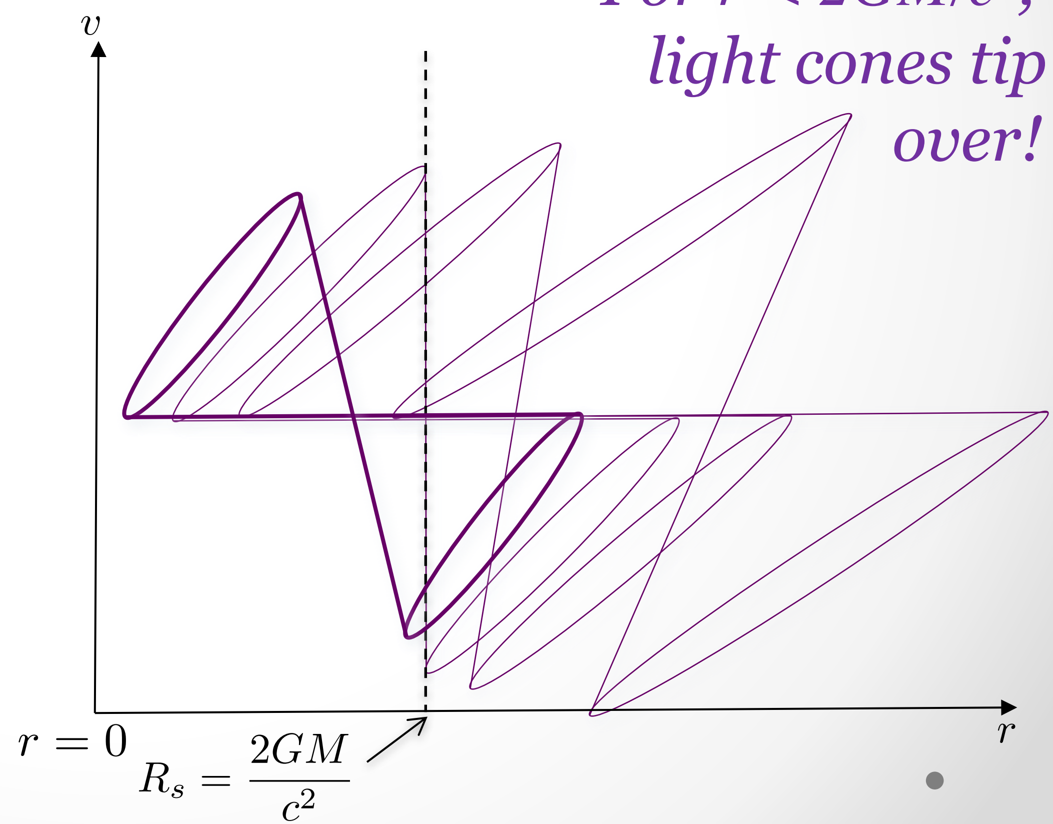
For $r < 2GM/c^2$, light cones tip over!



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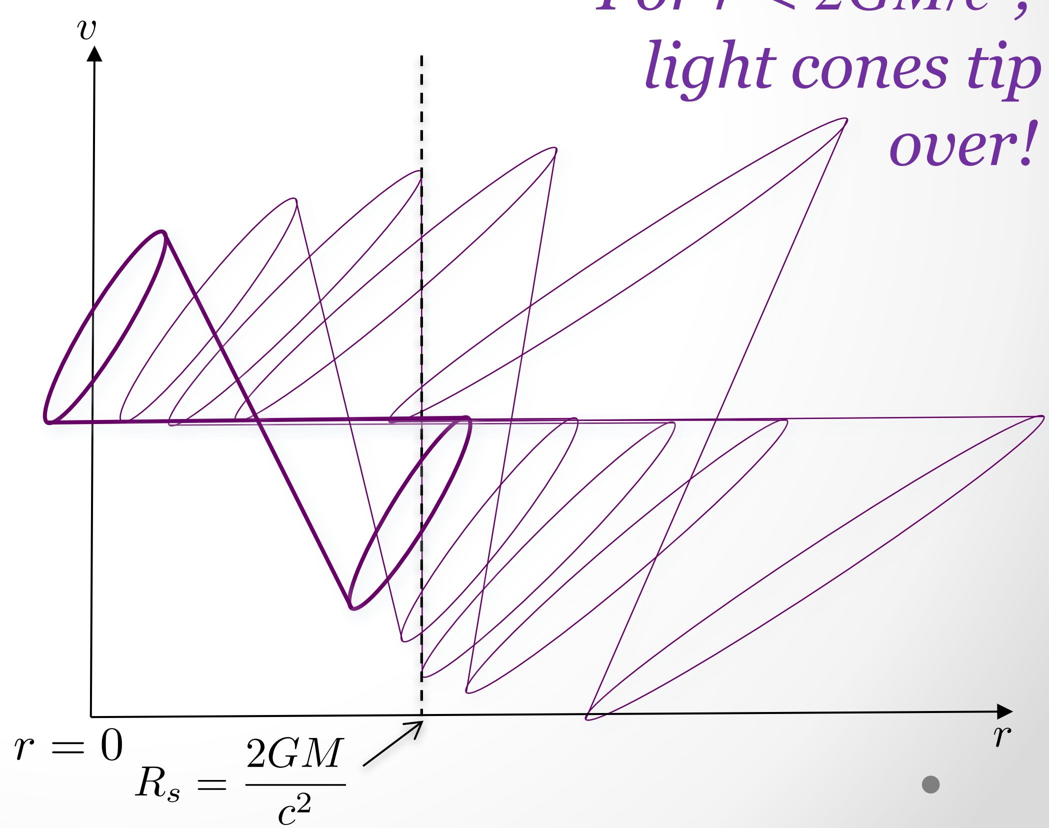

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In Eddington-Finkelstein Coordinates…

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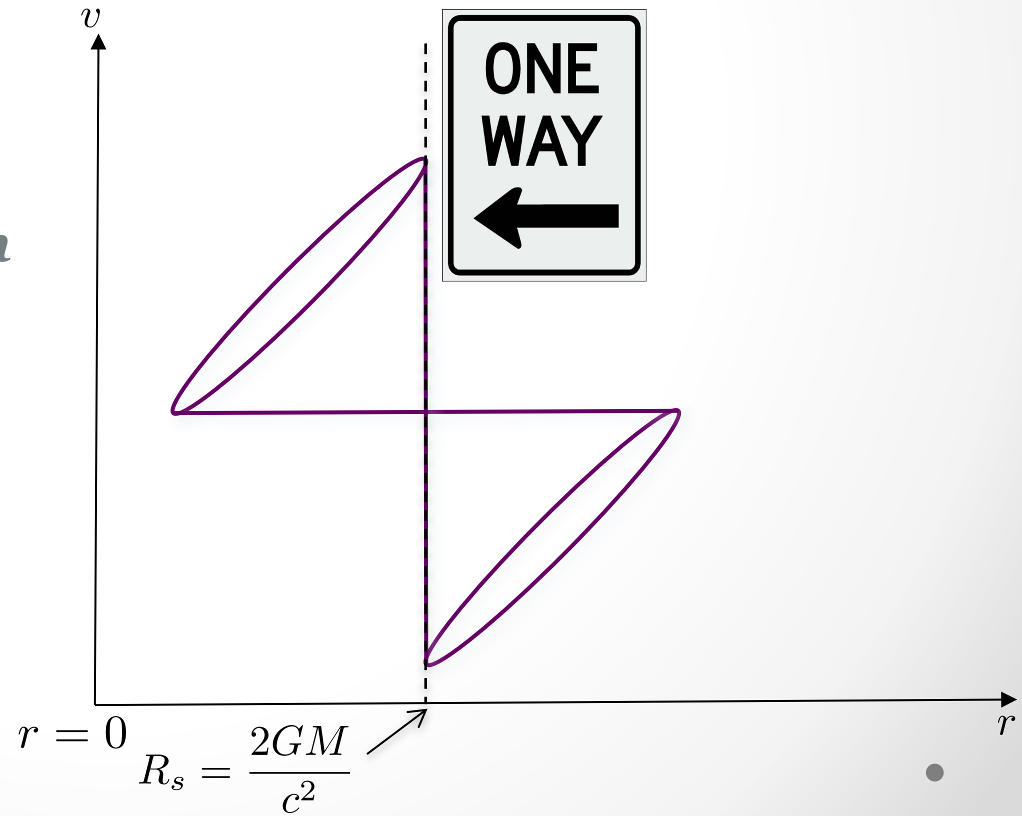
$$\frac{dv}{dr}\Big|_{r=R_s} = (0, \infty)$$

$r = 2GM/c^2$ is the ***Event Horizon***

$\Rightarrow$ *one-way surface*



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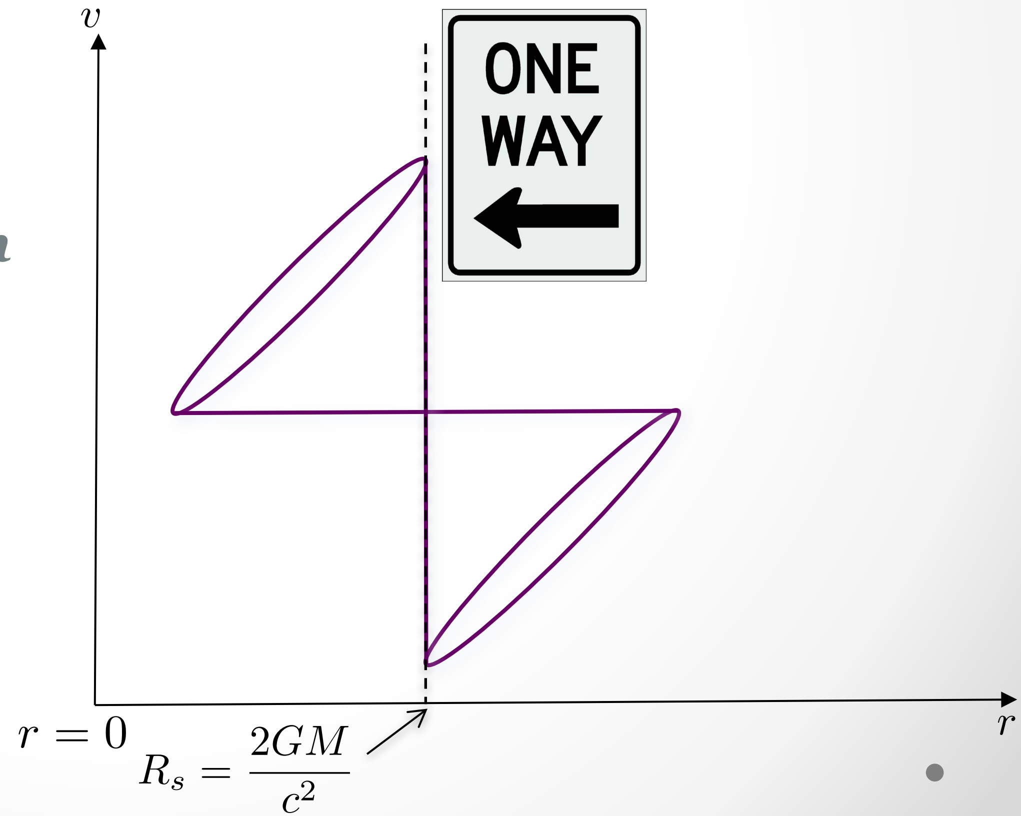
$$\frac{dv}{dr}\Big|_{r=R_s} = (0,\infty)$$

r = 2GM/c^2 is the ***Event Horizon***

$\Rightarrow$ *one-way surface*

For *r < 2GM/c*^{2}…
all future-directed paths are in direction of *decreasing r*!



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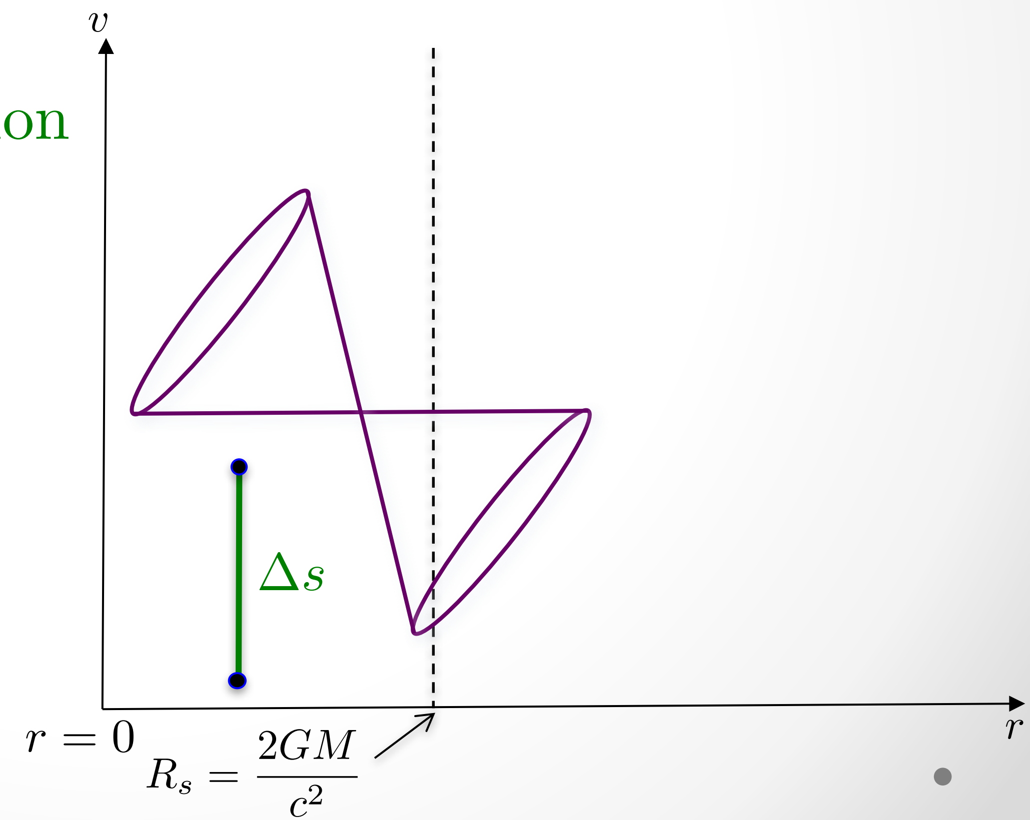
For $r < 2GM/c^2$, $r = const\ldots$

$ds^2 > 0$: spacelike separation



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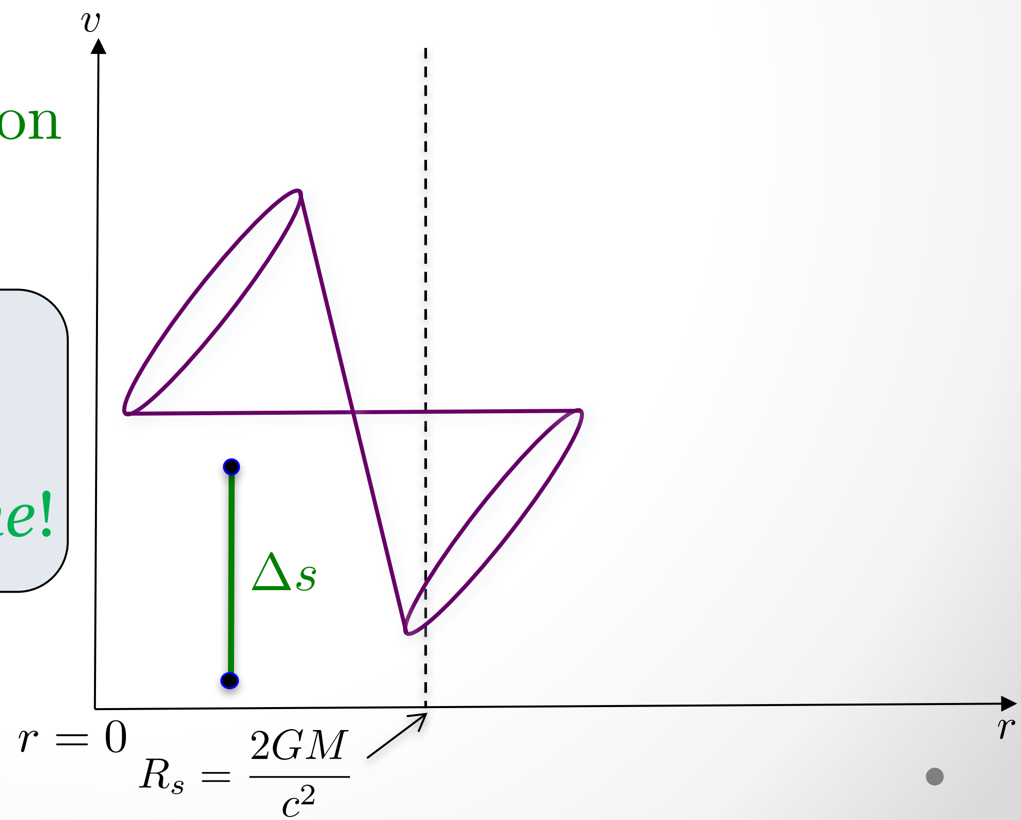
For $r < 2GM/c^2$, $r = const$…

$ds^2 > 0 \quad : \text{spacelike separation}$

The *r = 0* singularity is NOT a *position* in *space*, but rather a *moment* in *time*!



Conclusions

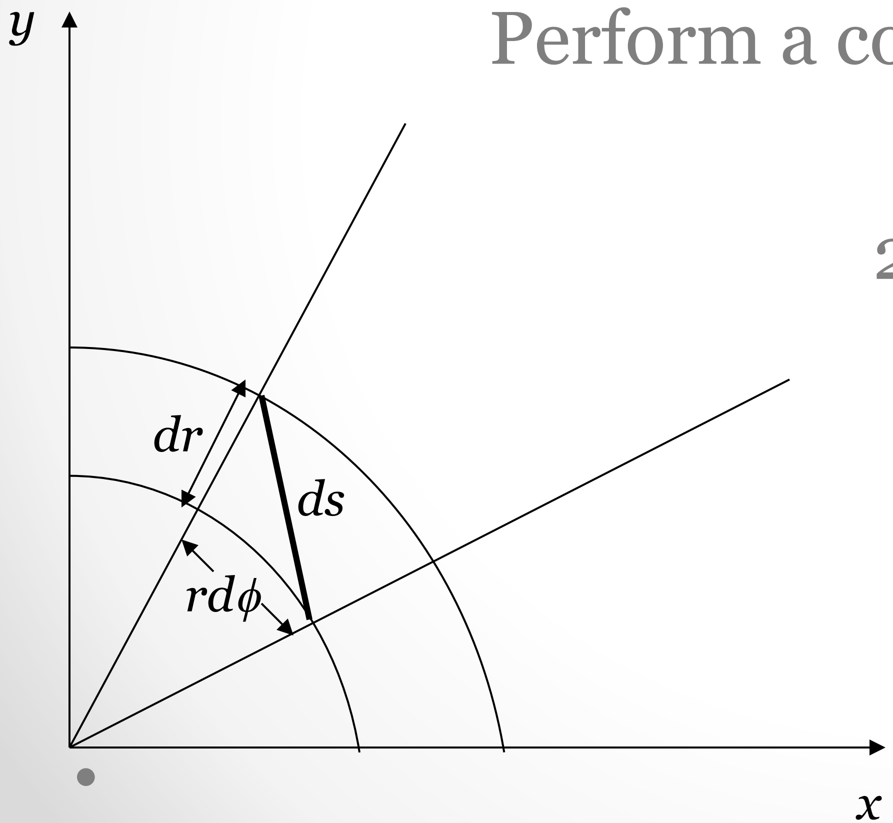
Classical general relativity predicts…

- the ultimate collapse of sufficiently massive stars to *black holes*
- a radially infalling observer reaches the $r = 0$ singularity in a *finite* proper time interval
- an $r = 0$ 'moment in time' singularity

Coordinate Singularity i.e.

2D Line element in polar coordinates..

$$ds^2 = dr^2 + r^2 d\phi^2$$



Perform a coordinate transformation...

$$r = a^2 / r'$$

2D Line element becomes..

$$ds^2 = \frac{a^4}{r'^4}\left(dr'^2 + r'^2\, d\phi^2\right)$$

Notice:

$$\lim_{r' \to 0} ds^2 \to \infty$$

Einstein's theory of special relativity

Two Postulates:

1. The laws of physics are the *same* in all inertial reference frames
2. The speed of light in a vacuum is equal to the value c, *independent* of the source.

Einstein's theory of special relativity

Two Postulates:

1. The laws of physics are the *same* in all inertial reference frames
2. The speed of light in a vacuum is equal to the value c, *independent* of the source.

Consequences:

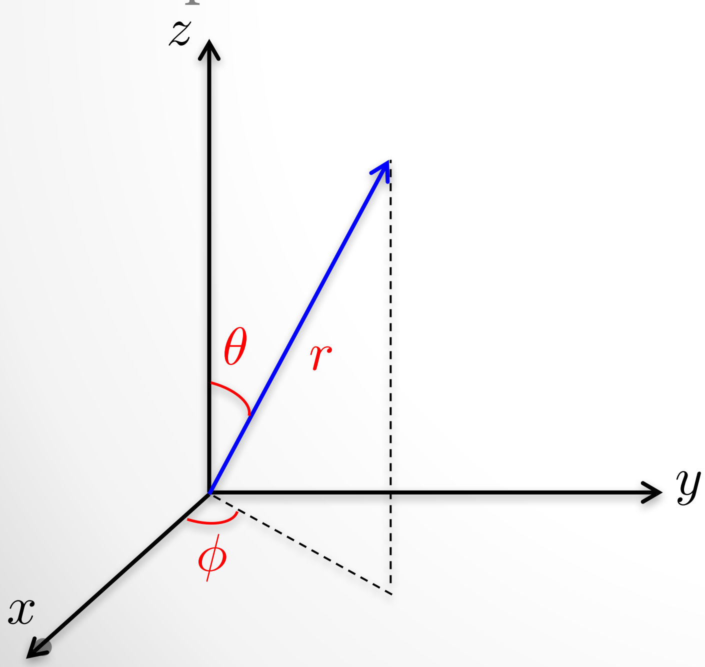
1. Time dilation
2. Length contraction
3. Relativity of simultaneity

Line elements are *invariant* under coordinate transformations…

The flat line element in *Cartesian coordinates*

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

perform a coordinate transformation to *spherical coordinates*

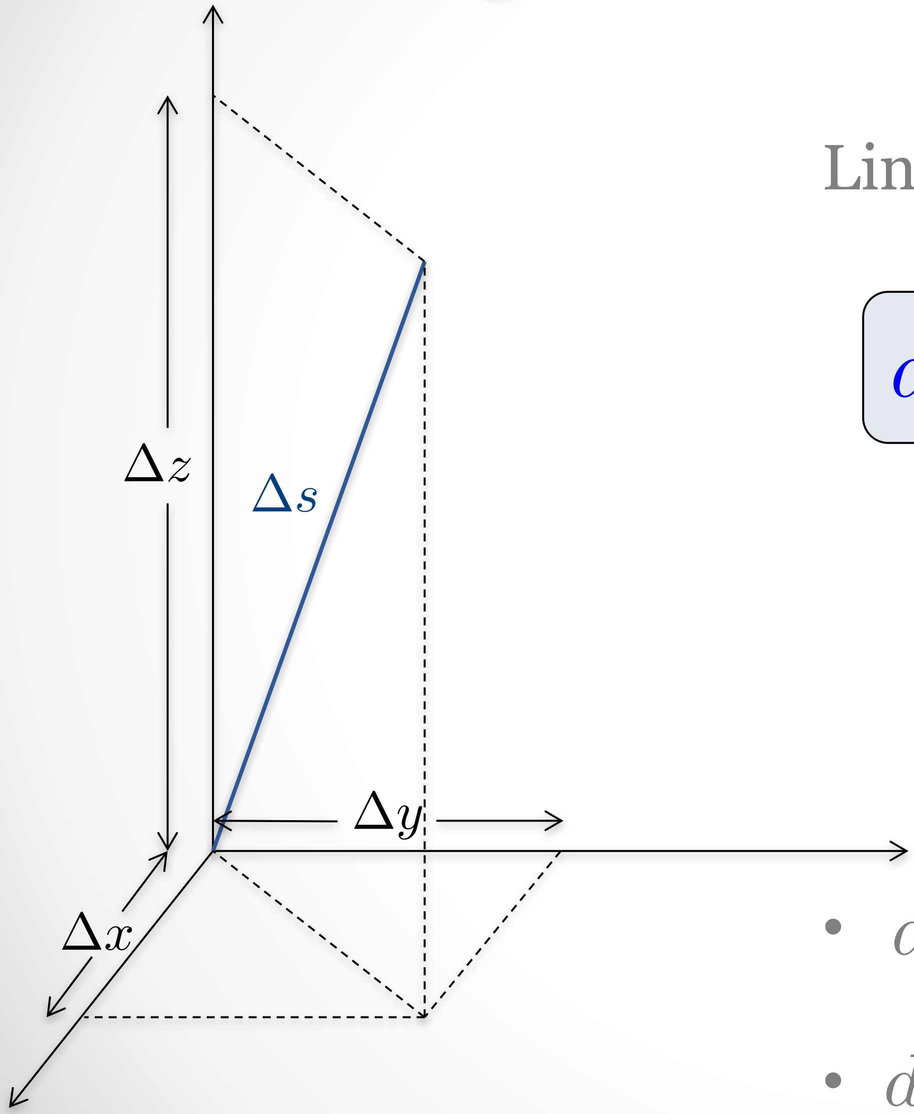


$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$

Same spacetime '*length*',
only coordinates have changed!

$\Rightarrow$ GR is *coordinate independent*

3D Euclidean Space



Line element in Euclidean space…

$$ds^2 = dx^2 + dy^2 + dz^2$$

- ds^2 is the *line element* measuring length
- ds^2 is *invariant* under rotations