Newtonian and general relativistic orbits with small eccentricities on 2D surfaces

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CMU Physics Seminar
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Outline

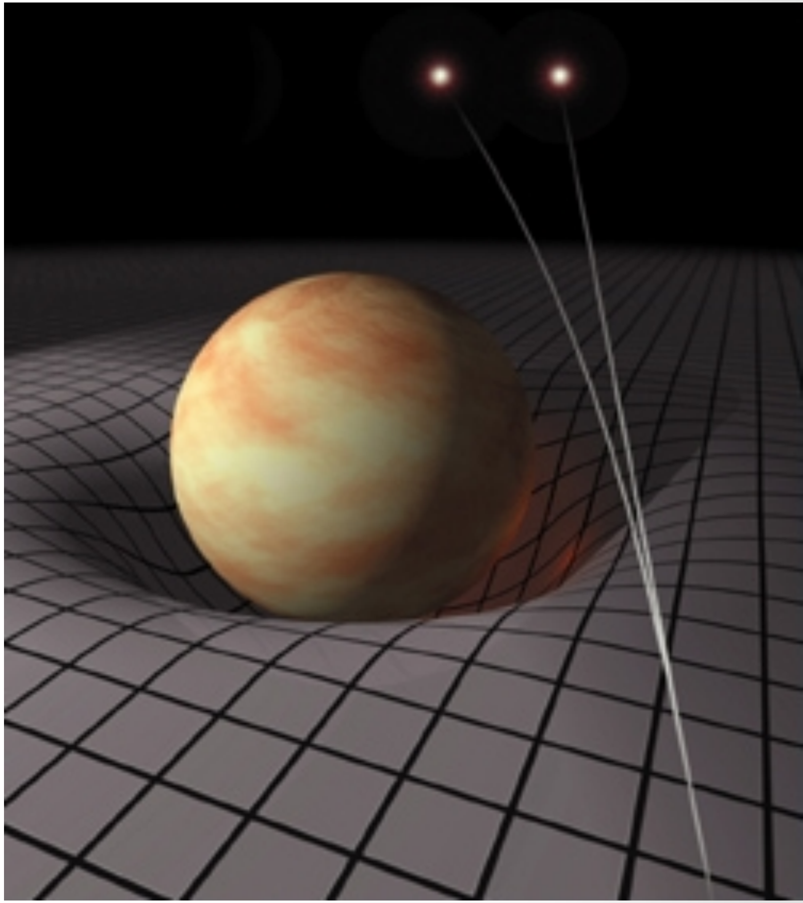
- The elliptical orbits of Newtonian gravitation
- The 2D surfaces that generate Newtonian orbits with small eccentricities
- Precessing elliptical orbits of GR with small eccentricities
- The 2D surfaces that generate general relativistic orbits with small eccentricities

Einstein's theory of general relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $G_{\mu\nu}$ describes the *curvature of spacetime*
- $T_{\mu\nu}$ describes the *matter & energy in spacetime*

Matter tells *space* how to *curve*,
space tells *matter* how to *move*.



Sean M. Carrol, *Spacetime and Geometry: An Introduction to Einstein's General Relativity* (Addison Wesley, 2004)

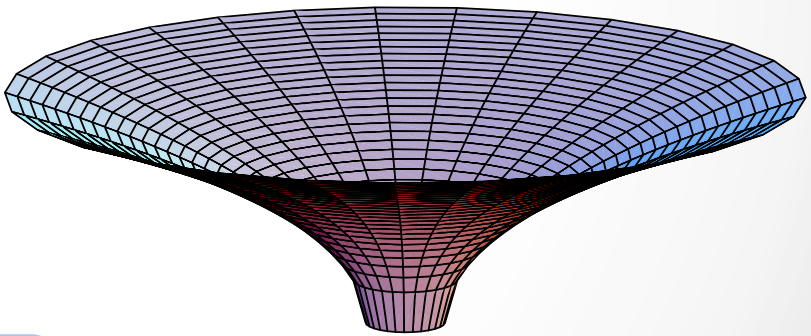
Einstein's theory of general relativity

Consider a *spherically-symmetric, non-rotating massive object…*

Embedding diagram ($t = t_0$, $\theta = \pi/2$)..

- 2D equatorial 'slice' of the 3D space at one moment in time

$$z(r) = 2\sqrt{\frac{2GM}{c^2}\left(r - \frac{2GM}{c^2}\right)}$$



where $2GM/c^2 = 1$

Is there a warped 2D surface that will yield the orbits of planetary motion?

The Lagrangian in spherical-polar coordinates with a Newtonian potential

- is of the form..

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GmM}{r}$$

- where we set $\theta = \pi/2$ and a dot refers to a time derivative.

- For the azimuthal-coordinate..

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$ yields $r^2\dot{\phi} = \ell$ -Kepler's 2nd Law

The Lagrange equations of motion

- For the radial-coordinate..

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

- yields the equation of motion for an object of mass *m*..

$$* \quad \ddot{r} - \frac{\ell^2}{r^3} + \frac{GM}{r^2} = 0$$

- Using the differential operator..

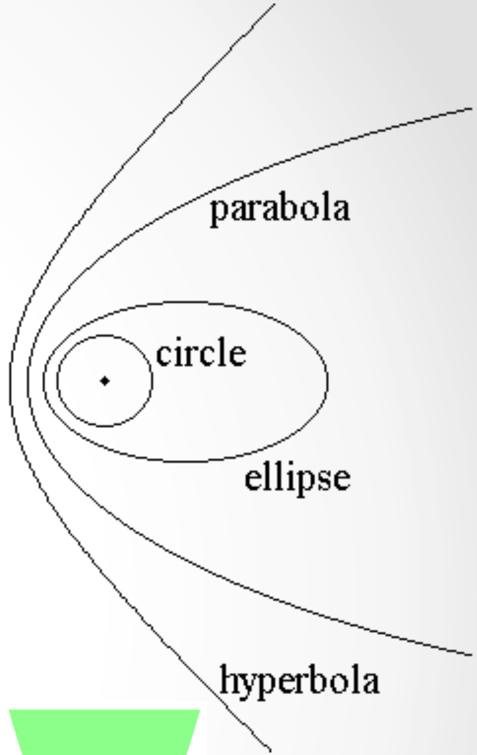
$$\frac{d}{dt} = \frac{\ell}{r^2}\frac{d}{d\phi} \quad ,$$

* can be written in the form..

The equation of motion

- For the radial-coordinate..

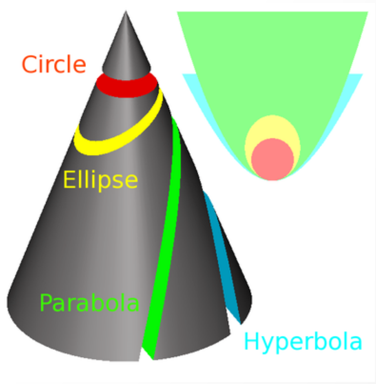
$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r + \frac{GM}{\ell^2}r^2 = 0$$



- yields the *conic sections*..

$$r(\phi) = \frac{r_0}{(1 + \varepsilon\cos\phi)}$$

-Kepler's 1st Law



- where $\ell^2 = GMr_0$
- and ε is the *eccentricity* of the orbit.

http://www.controlbooth.com/threads/cyc-color-wash-using-fresnels.30704/
http://conic-sections-section.wikispaces.com/General+Infomation+Main

The radial equation of motion

- the *exact* solution..

$$r(\phi)_{ex} = \frac{r_0}{(1 + \varepsilon \cos \phi)}$$

- for *small* eccentricities..

$$r(\phi)_{app} \simeq r_0(1 - \varepsilon \cos \phi) \quad *$$

Planets	r_o (m)	ε	% error
Mercury	$5.79*10^{10}$	0.2056	4.227
Venus	$1.08*10^{10}$	0.0068	0.005
Earth	$1.50*10^{11}$	0.0167	0.028
Mars	$2.28*10^{11}$	0.0934	0.872
Jupiter	$7.78*10^{11}$	0.0483	0.233
Saturn	$1.43*10^{12}$	0.056	0.314
Uranus	$2.87*10^{12}$	0.0461	0.213
Neptune	$4.50*10^{12}$	0.01	0.01

$$\% \text{ error} = \frac{|r_{ex} - r_{app}|}{r_{ex}} * 100\%$$

$$= \varepsilon^2 \cos \phi * 100\%$$

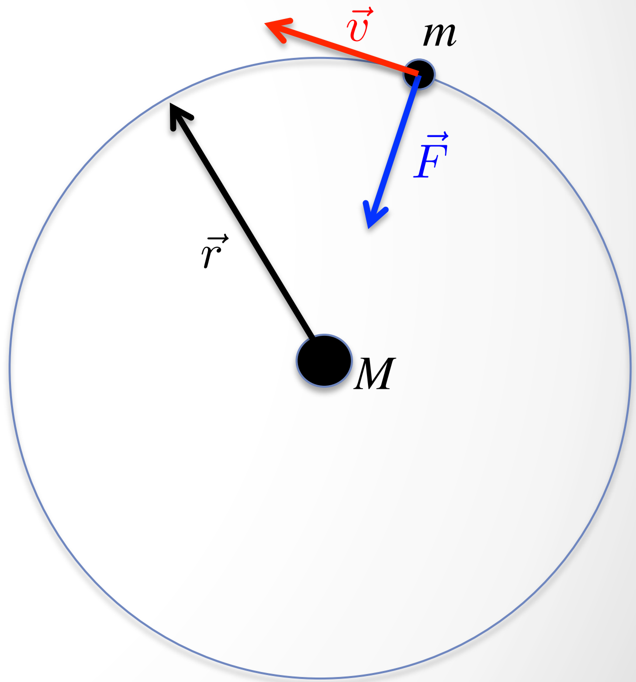
Notice:

- * yields an *excellent* approximation for the solar system planets!

Kepler's 3rd Law

- Setting $\dot{r} = \ddot{r} = 0$ and using $\dot{\phi} = 2\pi/T$ for circular orbits…

$$T^2 = \left(\frac{4\pi^2}{G}\right) \cdot \frac{r^3}{M}$$



Notice:

- Kepler's 3rd Law is *independent* of m!

An object orbiting on a 2D cylindrically-symmetric surface

- is described by a Lagrangian of the form..

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}I\omega^2 - mgz$$

- Now, for the orbiting object..

$$I = \alpha m R^2 \text{ and } \omega^2 = v^2/R^2 \qquad \text{so} \quad \frac{1}{2}I\omega^2 = \frac{1}{2}\alpha m v^2$$

where $\alpha = 2/5$ for a *rolling sphere*,
$\alpha = 0$ for a *sliding* object.

- The orbiting object is constrained to reside on the surface..

$$z = z(r)$$

The Lagrange equations of motion

- For the azimuthal-coordinate..

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \text{yields} \quad r^2\dot{\phi} = \ell/(1+\alpha)$$

- For the radial-coordinate..

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

- yields the equation of motion for the orbiting object..

$$(1+z'^2)\ddot{r} + z'z''\dot{r}^2 - \frac{\tilde{\ell}^2}{r^3} + \tilde{g}z' = 0 \quad \text{where} \quad \begin{array}{l} \tilde{\ell} \equiv \ell/(1+\alpha) \\ \tilde{g} \equiv g/(1+\alpha) \end{array}$$

The Lagrange equation of motion

Compare the equation of motion for the *orbiting object*..

$$(1 + z'^2)\ddot{r} + z'z''\dot{r}^2 - \frac{\tilde{\ell}^2}{r^3} + \tilde{g}z' = 0 \quad *$$

- to the equation of motion for *planetary orbits*..

$$\ddot{r} - \frac{\ell^2}{r^3} + \frac{GM}{r^2} = 0$$

* will NOT yield *Newtonian orbits* on ANY cylindrically-symmetric surface, *except* in the special case of circular orbits.

L.Q. English and A. Mareno, "Trajectories of rolling marbles on various funnels", Am. J. Phys. **80** (11), 996-1000 (2012)

The Lagrange equation of motion

Compare the equation of motion for the *orbiting object*..

$$(1 + z'^2)\ddot{r} + z'z''\dot{r}^2 - \frac{\tilde{\ell}^2}{r^3} + \tilde{g}z' = 0 \quad ^*$$

- to the equation of motion for *planetary orbits*..

$$\ddot{r} - \frac{\ell^2}{r^3} + \frac{GM}{r^2} = 0$$

* will NOT yield *Newtonian orbits* on ANY cylindrically-symmetric surface, *except* in the special case of circular orbits.

- So, what about for *nearly* circular orbits?

The radial equation of motion

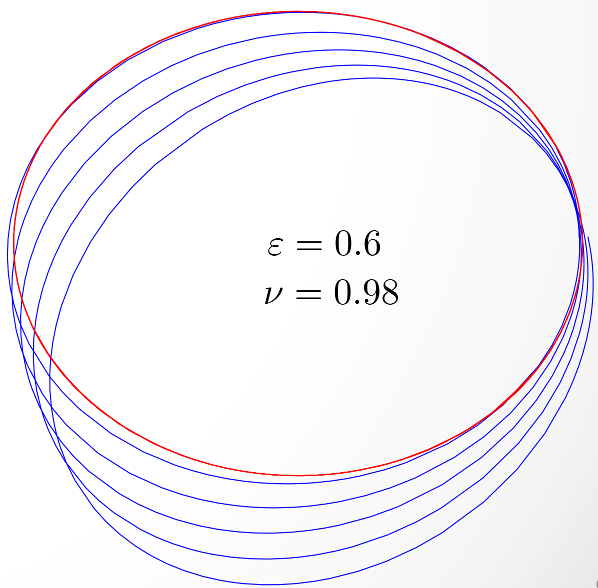
- Using the differential operator, the radial equation becomes..

$$(1+z'^2)\frac{d^2r}{d\phi^2} + \left[z'z'' - \frac{2}{r}(1+z'^2)\right]\left(\frac{dr}{d\phi}\right)^2 - r + \frac{\tilde{g}}{\tilde{\ell}^2}z'r^4 = 0$$

- For *nearly* circular orbits with small eccentricities..

$$r(\phi) = r_0(1 - \varepsilon\cos(\nu\phi))$$

where r_0 & ν are parameters.



Notice:

- when $\nu = 1$, *stationary* elliptical orbits
- when $\nu \neq 1$, *precessing* elliptical orbits.

The radial equation of motion

- We find the solution, to 1st order in the eccentricity, when..

$$\tilde{\ell}^2 = \tilde{g} r_0^3 z_0'$$
$$z_0'(1 + z_0'^2)\nu^2 = 3z_0' + r z_0''$$

*

For $z(r) \propto -\frac{1}{r^n}$ ...

- *Precessing* elliptical orbits when $n < 2$
- *Stationary* elliptical orbits, for certain radii, when $n < 1$
- *No* stationary elliptical orbits when $n = 1$!

Michael Nauenberg, "Perturbation approximation for orbits in axially symmetric funnels", *to appear in* Am. J. Phys.

The 2D surface that generates Newtonian orbits

- To find the 2D surface that yields *stationary* elliptical orbits with small eccentricities for *all* radii, solve…

$$z'\left(1+z'^2\right)=3z'+rz''$$

- The solution for the *slope* of the surface is..

$$\frac{dz}{dr}=\sqrt{2}\left(1+\kappa r^4\right)^{-1/2} \quad *$$

where κ is an *arbitrary* integration constant

Notice:

- * is *independent* of spin of orbiting object.
- When $\kappa=0$, * becomes the equation of an *inverted cone* with slope $\sqrt{2}$.

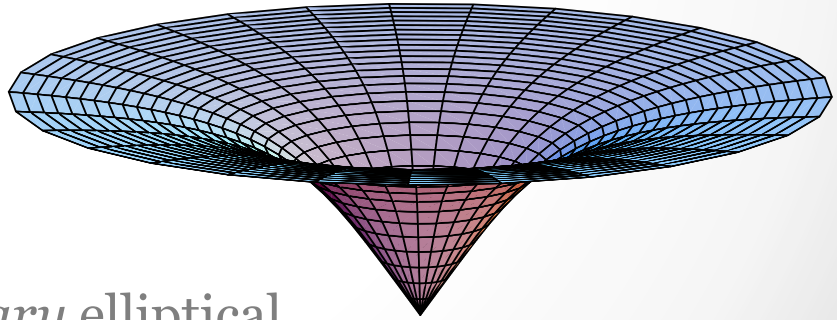
Gary D. White, "On trajectories of rolling marbles in cones and other funnels", Am. J. Phys. **81** (12), 890-898 (2013)

The 2D surface that generates Newtonian orbits

- Integrating yields the shape function…

$$z(r) = -\sqrt{2}\left(-\frac{1}{\kappa}\right)^{1/4} F(\sin^{-1}(-\kappa r^4)^{1/4}, -1)$$

- where *F(a,b)* is an *elliptic integral of the 1*st *kind.*



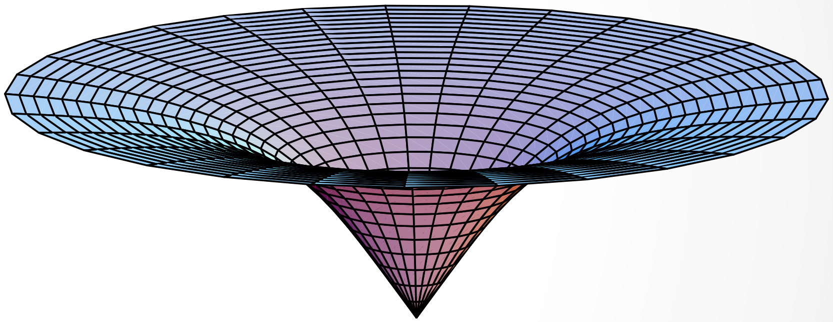
Notice:

- This 2D surface will generate *stationary* elliptical orbits with small eccentricities for *all* radii!

Kepler's 3rd Law

- Setting $\dot{r} = \ddot{r} = 0$ and using $\dot{\phi} = 2\pi/T$ for circular orbits…

$$T^2 = \left(\frac{2\sqrt{2}\pi^2}{\tilde{g}}\right) r\sqrt{1+\kappa r^4}$$



- Notice that when $\kappa r^4 \ll 1$ ..

$T^2 \propto r$ - Kepler-like relation for that of an *inverted cone.*

- and when $\kappa r^4 \gg 1$ ..

$T^2 \propto r^3$ - Kepler's 3rd Law of *planetary motion.*

Precessing elliptical orbits in GR with small eccentricities

- The eqn of motion for an object orbiting about a *non-rotating, spherically-symmetric object* of mass *M* in GR is..

$$\ddot{r} - \frac{\ell^2}{r^3} + \frac{GM}{r^2} + \frac{3GM\ell^2}{c^2 r^4} = 0 \quad *$$

- where a dot refers to a derivative w.r.t. *proper time*.
- Using the differential operator, * becomes..

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r + \frac{GM}{\ell^2} r^2 + \frac{3GM}{c^2} = 0$$

- we choose a solution of the form..

$$r(\phi) = r_0(1 - \varepsilon \cos(\nu\phi))$$ where r_0 & ν are parameters.

Precessing elliptical orbits in GR with small eccentricities

- We find the solution, to 1st order in the eccentricity, when..

$$\ell^2 = GMr_0 \left(1 - \frac{3GM}{c^2 r_0}\right)^{-1}$$

$$\nu^2 = 1 - \frac{6GM}{c^2 r_0}$$

Planets	r_o (m)	ε	$6GM/c^2r_o$
Mercury	$5.79*10^{10}$	0.2056	$1.53*10^{-7}$
Venus	$1.08*10^{10}$	0.0068	$8.19*10^{-7}$
Earth	$1.50*10^{11}$	0.0167	$5.90*10^{-8}$
Mars	$2.28*10^{11}$	0.0934	$3.88*10^{-8}$
Jupiter	$7.78*10^{11}$	0.0483	$1.14*10^{-8}$
Saturn	$1.43*10^{12}$	0.056	$6.19*10^{-9}$
Uranus	$2.87*10^{12}$	0.0461	$3.08*10^{-9}$
Neptune	$4.50*10^{12}$	0.01	$1.97*10^{-9}$

Notice:

- Deviation from *closed* elliptical orbits *increases* with *decreasing* r_0 .
- When $r_0 < 6GM/c^2$, ν becomes complex: *elliptical* orbits not allowed
- When $r_0 < 3GM/c^2$, ν & ℓ become complex: *no circular orbits*.

The 2D surfaces that generates general relativistic orbits

- To find the 2D surface that yields *precessing* elliptical orbits with small eccentricities for *all* radii, solve…

$$z'(1+z'^2)\nu = 3z' + rz'' \quad \text{with} \quad \nu = \sqrt{1-\frac{6GM}{c^2 r_0}}$$

- The solution for the *slope* of the surface is..

$$\frac{dz}{dr} = \sqrt{\frac{2+\beta}{1-\beta}} \cdot \left(1+\kappa r^{2(2+\beta)}\right)^{-1/2} \quad \text{where } \beta \equiv 6GM/c^2 r_0$$

Notice:

- dependent on *central mass*, *M*, and *average radius* of orbit, r_o.
- depends on β in both overall factor and in the power.
- Slope diverges as $\beta \to 1$

Compare the 2D surfaces…

- Slope that generates Newtonian *stationary* elliptical orbits..

$$\frac{dz}{dr} = \sqrt{2}\,(1+\kappa r^4)^{-1/2}$$

- Slope that generates the GR *precessing* elliptical orbits…

$$\frac{dz}{dr} = \sqrt{\frac{2+\beta}{1-\beta}} \cdot (1+\kappa r^{2(2+\beta)})^{-1/2}$$

where $\beta \equiv 6GM/c^2 r_0$

Planets	r_o (m)	ε	β
Mercury	$5.79*10^{10}$	0.2056	$1.53*10^{-7}$
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Notice:

- They agree when $\beta \to 0$.
- GR offers a *tiny* correction for the orbits of the solar system planets.