The Physics of Rainbows

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Quiz: The Physics of Rainbows

1. For the primary rainbow, what is the ‘exterior’ (largest observed angle) color?
   a) Violet    b) Red    c) Yellow    d) Green

2. For the primary rainbow, what is the ‘interior’ (smallest observed angle) color?
   a) Violet    b) Red    c) Yellow    d) Green

3. For the secondary rainbow, what is the ‘exterior’ color?
   a) Violet    b) Red    c) Yellow    d) Green

4. For the secondary rainbow, what is the ‘interior’ color?
   a) Violet    b) Red    c) Yellow    d) Green

5. Is the region between the primary and secondary rainbows dark or bright?

6. Is the interior region of the primary rainbow dark or bright?
What path should the lifeguard take to minimize her transit time?

\[ v_i > v_t \]
Smallest time interval?

Shortest distance = smallest time?

$v_i > v_t$
Smallest time interval?

Shortest $H_2O$ distance = smallest time?

$v_i > v_t$
Smallest time interval?

Shortest land distance = smallest time?
Smallest time interval?

Path of smallest time!

\( \nu_i > \nu_t \)
Smallest time interval?

\[ v_i > v_t \]
Smallest time interval?

\[ v_i > v_t \]
Smallest time interval?

\[ \sqrt{a^2 + x^2} \]

\[ \sqrt{b^2 + (c - x)^2} \]
To calculate the total time...

\[ t(x) = t_i + t_t \]

\[ = \frac{\sqrt{a^2 + x^2}}{v_i} + \frac{\sqrt{b^2 + (c - x)^2}}{v_t} \]
Plot of $t(x)$ vs $x$...

The total time is...

$$t(x) = t_i + t_t$$

$$= \frac{\sqrt{a^2 + x^2}}{v_i} + \frac{\sqrt{b^2 + (c - x)^2}}{v_t}$$

$$a, b, c = 1$$

$$v_i = 0.8$$

$$v_t = 0.6$$
Smallest time interval?

To calculate the path of smallest time...

\[
\frac{d}{dx} t(x) = 0 \quad \text{yields}
\]

\[
\frac{1}{v_i} \frac{x}{\sqrt{a^2 + x^2}} = \frac{1}{v_t} \frac{(c - x)}{\sqrt{b^2 + (c - x)^2}}
\]

\[v_i > v_t\]
The path of smallest time... 

\[ n_i \sin \theta_i = n_t \sin \theta_t \]

where \( n \equiv \frac{c}{v} \)

\( n_i < n_t \)
**Fermat’s Principle:**

The actual path between two points taken by a beam of light is the one that is traversed in the *least time*.

When light enters a new medium, it’s path obeys Snell’s Law:

\[ n_i \sin \theta_i = n_t \sin \theta_t \]
- Rays of sunlight are *nearly parallel* to each other.

- Suspended H$_2$O droplets are *nearly spherical* due to *surface tension*.

- $\theta_d$ is the *deviation angle*.
- $\alpha$ is the *observed angle*, measured from the *anti-solar* direction.
At certain observed angles, a particular color will dominate.

- This angle forms a cone around the anti-solar direction.
- All raindrops that lie on this cone can contribute to the rainbow

=> The rainbow is circular!
Primary Rainbow:

Light is incident at angle $\theta_i$ ...

1st refraction...
- transmitted angle given by Snell’s Law

$$\theta_t = \sin^{-1} \left( \frac{\sin \theta_i}{n} \right)$$

where

$$n \equiv \frac{n_t}{n_i}$$

$n_{air} \approx 1.00$

$n_{H2O} \approx 1.33$
Primary Rainbow:

1st refraction,
- by geometry, same angle
Primary Rainbow:

1st refraction, 2nd reflection...

- Law of Reflection, same angle
Primary Rainbow:

1st refraction,
2nd reflection...

- Law of Reflection, same angle
- by geometry, same angle
Primary Rainbow:

1\textsuperscript{st} refraction,
2\textsuperscript{nd} reflection,
3\textsuperscript{rd} refraction...
• by Snell’s Law, same $\theta_i$
Calculate the net deviation angle, $\theta_d$ ...

\[ \theta_d = (\theta_i - \theta_t) + \]

$\theta_i$

$\theta_t$

$\theta_d$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_t$

$\theta_i$

$\theta_i$

$\theta_i$

$\theta_i$

$\theta_i$

$\theta_i$

$\theta_i$

$\theta_i$

$\theta_i$
Calculate the *net deviation angle*, $\theta_d$ ...

\[ \theta_d = (\theta_i - \theta_t) + (180^\circ - 2\theta_t) + \theta_t \]
Calculate the *net deviation angle*, $\theta_d$...

$$\theta_d = (\theta_i - \theta_t) + (180^\circ - 2\theta_t) + (\theta_i - \theta_t)$$
The net deviation angle, $\theta_d$, is

$$\theta_d = 180^\circ + 2\theta_i - 4\theta_t$$

or

$$\theta_d(\theta_i) = 180^\circ + 2\theta_i - 4 \sin^{-1}\left(\frac{\sin \theta_i}{n}\right)$$

where we used Snell’s Law.

$$\alpha = 180^\circ - \theta_d$$

observed angle
An infinite number of parallel sunbeams hit the spherical raindrop, so which ones do we see?

Notice:
The ‘Cartesian Ray’ is the ray that has the minimum deviation angle.
Plot of $\theta_d$ vs $\sin^{-1}\theta_i$...

The backscattered rays cluster at the minimum deviation angle, yielding an enhanced brightness.

The net deviation angle is...

$$\theta_d = 180^\circ + 2\theta_i - 4\sin^{-1}\left(\frac{\sin\theta_i}{n}\right)$$
To calculate the minimum deviation angle...

\[ \frac{d\theta_d}{d\theta_i} = 0 \quad \text{yields} \]

\[ \sin \theta_i = \sqrt{\frac{4 - n^2}{3}} \]
Dispersion...

Different frequencies of light have different indices of refraction.

\[ n_{t,r} = 1.331 \]
\[ n_{t,v} = 1.344 \]

\[ \alpha_r = 42.4^\circ \]
\[ \alpha_v = 40.5^\circ \]
\[ \Delta \alpha = 1.9^\circ \]

\[ \alpha(n) = -2 \sin^{-1} \left( \frac{\sqrt{4 - n^2}}{3} \right) + 4 \sin^{-1} \left( \frac{\sqrt{4 - n^2}}{3n^2} \right) \]

Notice:
The ‘outside’ of the primary rainbow is red, whereas the ‘inside’ is violet!
\[ \theta_d = (\theta_i - \theta_t) + (180^\circ - 2\theta_t) + (\theta_i - \theta_t) \]
The *net deviation angle*, $\theta_d$, is...

$$\theta_d = 360^\circ + 2\theta_i - 6\theta_t$$

or

$$\theta_d(\theta_i) = 360^\circ + 2\theta_i - 6 \sin^{-1} \left( \frac{\sin \theta_i}{n} \right)$$

where we again used Snell’s Law.

\[\alpha = \theta_d - 180^\circ\]

observed angle
Minimum deviation angle...

To calculate the minimum deviation angle...

\[
\frac{d\theta_d}{d\theta_i} = 0 \quad \text{yields}
\]

\[
\sin \theta_i = \sqrt{\frac{9 - n^2}{8}}
\]
Secondary Rainbow...

\[ n_{t,r} = 1.331 \]
\[ n_{t,v} = 1.344 \]

\[ \alpha_r = 50.3^\circ \]
\[ \alpha_v = 53.8^\circ \]
\[ \Delta \alpha = 3.5^\circ \]

\[ \alpha(n) = 180^\circ + 2 \sin^{-1}\left(\sqrt{\frac{9-n^2}{8}}\right) - 6 \sin^{-1}\left(\sqrt{\frac{9-n^2}{8n^2}}\right) \]

Notice:
The ‘outside’ of the secondary rainbow is violet, whereas the ‘inside’ is red!
Why is the interior region of the primary rainbow bright?

For the **primary rainbow**... one has backscattering for all angles in the regime:

\[ 0 \leq \alpha \leq 42.4^\circ \]

Your eye receives backscattered light of *all wavelengths* from the *interior* of the primary rainbow

\[ \Rightarrow \text{Bright white light!} \]
Why is the region between the primary and secondary rainbows dark?

For the primary rainbow... one has backscattering when
\[ 0 \leq \alpha \leq 42.4^\circ \]

For the secondary rainbow... one has backscattering when
\[ 50.3^\circ \leq \alpha \leq 180^\circ \]

One has ZERO scattering from one or two reflections when
\[ 42.4^\circ < \alpha < 50.3^\circ \]

=> Alexander’s dark band!
Is a 3\textsuperscript{rd} (or \(l\)\textsuperscript{th}) rainbow theoretically possible?

After allowing for \(l\) internal reflections, the \textit{net deviation angle} is...

\[
\theta_d(\theta_i) = l(180^\circ) + 2\theta_i - 2(l + 1) \sin^{-1}\left(\frac{\sin \theta_i}{n}\right)
\]

To calculate the \textit{minimum deviation angle}...

\[
\frac{d\theta_d}{d\theta_i} = 0 \quad \text{yields} \quad \sin \theta_i = \sqrt{\frac{(l + 1)^2 - n^2}{l(l + 2)}}
\]

⇒ The first 13 rainbows of water have been observed from a drop suspended in a spectrometer!*

Why are two rainbows sometimes visible in the sky, but one never sees a third (or fourth)?

For the tertiary rainbow \((l = 3)\)...

\[
\begin{align*}
\alpha_r &= 137.5^\circ \\
\alpha_v &= 144.3^\circ \\
\Delta\alpha &= 6.8^\circ
\end{align*}
\]

So why don’t you see it?

1. Each successive Cartesian ray is at a greater incident angle, therefore a reduction in intercepting cross-sectional area.
2. Larger \(\Delta\alpha\) for each successive rainbow.
3. Loss of light due at each successive reflection.