Einstein’s Classical Theory of General Relativity and Black Holes

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Outline...

- Length in 2D space
- Special relativity & the relativity of simultaneity
- ‘Length’ in 4D spacetime
- Spacetime diagrams & light cones
- The Schwarzschild solution & black holes
2D Euclidean Space

Line element in Euclidean space...

\[
\lim_{\Delta \to 0} \Delta s^2 = \Delta x^2 + \Delta y^2
\]

or

\[
ds^2 = dx^2 + dy^2
\]

• \(ds^2\) is the line element measuring length
Rotations in 2D Euclidean Space

Line element in Euclidean space...

\[ ds^2 = dx^2 + dy^2 = dx'^2 + dy'^2 = ds'^2 \]

- \( ds^2 \) is the line element measuring length
- \( ds^2 \) is invariant under rotations
Einstein’s theory of special relativity

Lorentz Boosts:

\[ ct' = \gamma \left( ct - \frac{v}{c} x \right) \]
\[ x' = \gamma \left( x - vt \right) \]
\[ y' = y \]
\[ z' = z \]

Time and 3D length depend on the frame of reference!
Relativity of Simultaneity

Notice:

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Notice:
Relativity of Simultaneity

Notice:

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Notice:

- $t'_{B} = t'_{A}$
Relativity of Simultaneity

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Relativity of Simultaneity

Notice:
- $t_B < t_A$
- $t'_B = t'_A$
- $t''_B > t''_A$
4D Minkowski Spacetime

Line element in Minkowski (flat) spacetime

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

- \( ds^2 \) is the line element measuring "length"
4D Minkowski Spacetime

Line element in Minkowski (flat) spacetime

\[ ds^2 = -c^2 dt^2 + dx^2 \]

- \( ds^2 \) is the line element measuring ‘length’
Consider a *primed* frame moving relative to the *unprimed* frame...
Lorentz Boosts in 4D Minkowski Spacetime

Line element in Minkowski (flat) spacetime

\[ ds^2 = -c^2 dt^2 + dx^2 \]

\[ = -c^2 dt'^2 + dx'^2 = ds'^2 \]

\( ds^2 \) is the line element measuring ‘length’

\( ds^2 \) is invariant under ‘rotations’
Consider radial null curves ($\theta & \phi = \text{const}, \ ds^2 = 0$)...

$$ds^2 = 0 = -c^2 dt^2 + dr^2$$

-> world line of a light beam!

$$c \frac{dt}{dr} = \pm 1$$

Particle's worldline
Consider radial null curves \((\theta & \phi = \text{const}, \; ds^2 = 0)\)...

\[
ds^2 = 0 = -c^2 dt^2 + dr^2
\]

\rightarrow \text{world line of a light beam!}

For 2 events..

\[
ds^2 = 0 : \text{lightlike separation}
\]
Consider *radial null curves* \((\theta \& \phi = \text{const}, \, ds^2 = 0)\)...

\[
ds^2 = 0 = -c^2 dt^2 + dr^2
\]

-> world line of a light beam!

For 2 events..

\[
ds^2 = 0 \quad : \text{lightlike separation}
\]

\[
ds^2 < 0 \quad : \text{timelike separation}
\]
Consider radial null curves \((\theta & \phi = \text{const}, \, ds^2 = 0)\)...

\[
ds^2 = 0 = -c^2 dt^2 + dr^2
\]

-> world line of a light beam!

\[
\frac{dt}{c dr} = \pm 1
\]

For 2 events..

\[
ds^2 = 0 \quad : \text{lightlike separation}
\]

\[
ds^2 < 0 \quad : \text{timelike separation}
\]

\[
ds^2 > 0 \quad : \text{spacelike separation}
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Consider *radial null curves* \((\theta \& \phi = \text{const}, \, ds^2 = 0)\)...

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For 2 events...

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    ds^2 = 0 : \text{lightlike separation}
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\[
    ds^2 < 0 : \text{timelike separation}
\]

\[
    ds^2 > 0 : \text{spacelike separation}
\]

*Massive particles follow *timelike world lines*!*
Lorentz Boosts in 4D Minkowski Spacetime

Consider 2 *timelike* separated events...

Notice:

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- 
-
Lorentz Boosts in 4D Minkowski Spacetime

Consider 2 timelike separated events...

Notice:
- $t_B > t_A$
- 
- 

Event $A$

$\Delta s$

Event $B$

$ct$ vs $x$
Consider 2 timelike separated events...

Notice:
- $t_B > t_A$
- $t'_B > t'_A$
Consider 2 timelike separated events...

Notice:

- $t_B > t_A$
- $t'_B > t'_A$
- $t''_B > t''_A$
Consider 2 timelike separated events...

Notice:
- $t_B > t_A$
- $t_B' > t_A'$
- $t_B'' > t_A''$

*Time ordering is preserved between timelike separated events!*
Consider 2 *spacelike* separated events...

Notice:
- $t_B > t_A$
- 
- 

**Lorentz Boosts in 4D Minkowski Spacetime**
Consider 2 \textit{spacelike} separated events...

Notice:
- $t_B > t_A$
- $t'_B = t'_A$
- \( \Delta s \)
Lorentz Boosts in 4D Minkowski Spacetime

Consider 2 spacelike separated events...

Notice:
- $t_B > t_A$
- $t'_B = t'_A$
- $t''_B < t''_A$
Lorentz Boosts in 4D Minkowski Spacetime

Consider 2 *spacelike* separated events...

Notice:
- $t_B > t_A$
- $t'_B = t'_A$
- $t''_B < t''_A$

*Time ordering is NOT preserved between spacelike separated events!*
Minkowski line element in spherical coordinates...

\[ ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \]

Notice:
- For constant time slice, spherical wave front
Spacetime diagram & light cones

Minkowski line element in spherical coordinates...

\[ ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \]

Notice:
- For constant time slice, spherical wave front
- **Light cone is a one-way surface**
Minkowski line element in spherical coordinates...

\[ ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \]

Notice:
- For constant time slice, spherical wave front
- **Light cone is a one-way surface**
- **Global** arrangement of light cones determine causal structure of spacetime
Einstein’s theory of general relativity

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

- \( G_{\mu\nu} \) describes the curvature of spacetime
- \( T_{\mu\nu} \) describes the matter & energy in spacetime

Matter tells space how to curve
Space tells matter how to move

The spherically-symmetric, time-independent, vacuum solution to GR is...

\[ ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2 d\Omega^2 \]

Let:

\[ f(r) \equiv 1 - \frac{2GM}{c^2r} \]

Notice:

\[ \lim_{r \to 2GM/c^2} f(r) \to 0 : \text{singularity} \]
\[ \lim_{r \to 0} f(r) \to \infty : \text{singularity} \]
Coordinate vs physical singularities...

Calculate an invariant scalar quantity:

\[ I \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12 \left( \frac{2GM}{c^2 r^3} \right)^2 \]

Notice:

\[ \lim_{r \to 2GM/c^2} I \to \frac{12}{R_s^4} : \text{well behaved (coordinate singularity)} \]

\[ \lim_{r \to 0} I \to \infty : \text{divergent (physical singularity)} \]
Coordinate vs physical singularities...

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The Schwarzschild coordinate system is pathological at \( r = 2GM/c^2 \) & MUST be abandoned there!
Black holes...

- For fixed radius $R$...

$$p(r = 0) \to \infty \quad \text{when} \quad M_{\text{max}} = \frac{4c^2}{9G} R$$

- For $M_{\text{star}} \sim 3-4 \, M_{\odot}$, star collapses to a black hole!

- Black holes have an event horizon (a.k.a. Schwarzschild radius):

$$R_s = \frac{2GM}{c^2}$$

- Black holes don’t suck!
  (External geometry of a black hole is the same as that of a star or planet)

Radial plunge of an *experimental* physicist.

\[
\tau = \tau_* - \frac{2}{3} \frac{R_s}{c} \left( \frac{r}{R_s} \right)^{3/2}
\]

\[
t = t_* + \frac{R_s}{c} \left[ -\frac{2}{3} \left( \frac{r}{R_s} \right)^{3/2} - 2 \left( \frac{r}{R_s} \right)^{1/2} + \ln \left[ \frac{\sqrt{r/R_s} + 1}{\sqrt{r/R_s} - 1} \right] \right]
\]

Notice:

\[
\lim_{r \to R_s} \tau \to \text{finite}
\]

\[
\lim_{r \to R_s} t \to \infty
\]

It takes a *finite* amount *proper time*, \(\tau\), to reach \(r = R_s\) but an *infinite* amount of *coordinate time*, \(t\)!
Consider radial null curves ($\theta & \phi = \text{const}, ds^2 = 0$)...

\[ ds^2 = 0 = - \left(1 - \frac{2GM}{c^2r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 \]

-> world line of a light beam!

\[ c \frac{dt}{dr} = \pm \frac{1}{\left(1 - \frac{2GM}{c^2r}\right)} \]

Notice:

\[ \lim_{r \to R_s} c \frac{dt}{dr} \to \pm \infty \]
Consider radial null curves ($\theta & \phi = \text{const}, \, ds^2 = 0$)...

$$ds^2 = 0 = - \left(1 - \frac{2GM}{c^2r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2$$

$\Rightarrow$ world line of a light beam!

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Consider radial null curves \((\theta \& \phi = \text{const}, ds^2 = 0)\)...

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\[ \rightarrow \text{world line of a light beam!} \]

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Consider radial null curves (θ & φ = const, \( ds^2 = 0 \))...

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ds^2 = 0 = - \left( 1 - \frac{2GM}{c^2r} \right) c^2 dt^2 + \left( 1 - \frac{2GM}{c^2r} \right)^{-1} dr^2
\]

-> world line of a light beam!

\[
\frac{dt}{c} \frac{1}{dr} = \pm \frac{1}{\left( 1 - \frac{2GM}{c^2r} \right)}
\]

Notice:

\[
\lim_{r \to R_s} c \frac{dt}{dr} \to \pm \infty
\]
Consider radial null curves ($\theta \& \phi = \text{const}, ds^2 = 0$)...

\[ ds^2 = 0 = - \left( 1 - \frac{2GM}{c^2r} \right) c^2 dt^2 + \left( 1 - \frac{2GM}{c^2r} \right)^{-1} dr^2 \]

-> world line of a light beam!

\[ \frac{dt}{c \frac{dr}{dr}} = \pm \frac{1}{(1 - \frac{2GM}{c^2r})} \]

Notice:

\[ \lim_{r \to R_s} c \frac{dt}{dr} \to \pm \infty \]

Light cones “close up” at $r = 2GM/c^2$!
In Eddington-Finkelstein Coordinates...

\[ ds^2 = 0 = - \left( 1 - \frac{2GM}{c^2 r} \right) dv^2 + 2dvdr \]

The slope of the light cone structure of spacetime is...

\[ \frac{dv}{dr} = 0 \text{ , (ingoing)} \]

\[ \frac{dv}{dr} = \frac{2}{\left( 1 - \frac{2GM}{c^2 r} \right)} \]

Notice:

\[ \frac{dv}{dr} \mid_{r > R_s} = (0, +) \]
In Eddington-Finkelstein Coordinates...

\[ ds^2 = 0 = - \left( 1 - \frac{2GM}{c^2r} \right) dv^2 + 2dvdr \]

The *slope* of the light cone structure of spacetime is...

\[ \frac{dv}{dr} = 0 \quad \text{(ingoing)} \]
\[ \frac{dv}{dr} = \frac{2}{\left( 1 - \frac{2GM}{c^2r} \right)} \]

Notice:

\[ \frac{dv}{dr} \bigg|_{r>Rs} = (0, +) \]
In Eddington-Finkelstein Coordinates...

\[ ds^2 = 0 = - \left( 1 - \frac{2GM}{c^2r} \right) dv^2 + 2dvdr \]

The slope of the light cone structure of spacetime is...

\[ \frac{dv}{dr} = 0 \quad (\text{ingoing}) \]

\[ \frac{dv}{dr} = \frac{2}{\left( 1 - \frac{2GM}{c^2r} \right)} \]

Notice:

\[ \frac{dv}{dr} \mid_{r > R_s} = (0, +) \]

\[ R_s = \frac{2GM}{c^2} \]
In Eddington-Finkelstein Coordinates...

\[ ds^2 = 0 = - \left( 1 - \frac{2GM}{c^2r} \right) dv^2 + 2dvdr \]

The slope of the light cone structure of spacetime is...

\[ \frac{dv}{dr} = 0 \text{, (ingoing)} \]
\[ \frac{dv}{dr} = \frac{2}{\left( 1 - \frac{2GM}{c^2r} \right)} \]

Notice:

\[ \frac{dv}{dr} \bigg|_{r>R_s} = (0, +) \]
\[ \frac{dv}{dr} \bigg|_{r<R_s} = (0, -) \]

For \( r < 2GM/c^2 \), light cones tip over!
In Eddington-Finkelstein Coordinates...

\[ ds^2 = 0 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 + 2dvdr \]

The slope of the light cone structure of spacetime is...

\[ \frac{dv}{dr} = 0 \quad \text{(ingoing)} \]
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For \( r < \frac{2GM}{c^2} \), light cones tip over!
In Eddington-Finkelstein Coordinates...

Notice:

\[ \frac{dv}{dr} \bigg|_{r=R_s} = (0, \infty) \]

\( r = \frac{2GM}{c^2} \) is the **Event Horizon**

⇒ one-way surface
In Eddington-Finkelstein Coordinates...

Notice:

\[ \frac{dv}{dr} \bigg|_{r=R_s} = (0, \infty) \]

\( r = \frac{2GM}{c^2} \) is the **Event Horizon**
\( \Rightarrow \) one-way surface

For \( r < \frac{2GM}{c^2} \)... all future-directed paths are in direction of decreasing \( r \)!
In Eddington-Finkelstein Coordinates...

\[ ds^2 = - \left( 1 - \frac{2GM}{c^2 r} \right) dv^2 + 2dvdr + r^2 d\Omega^2 \]

For \( r < \frac{2GM}{c^2} \), \( r = \text{const} \)...

\( ds^2 > 0 \) : spacelike separation
In Eddington-Finkelstein Coordinates...

\[ ds^2 = - \left( 1 - \frac{2GM}{c^2 r} \right) dv^2 + 2dvdr + r^2 d\Omega^2 \]

For \( r < 2GM/c^2 \), \( r = \text{const} \)...

\( ds^2 > 0 \) : spacelike separation

- The \( r = 0 \) singularity is NOT a position in space, but rather a moment in time!
Conclusions

Classical general relativity predicts...

• the ultimate collapse of sufficiently massive stars to black holes

• a radially infalling observer reaches the $r = 0$ singularity in a finite proper time interval

• an $r = 0$ ‘moment in time’ singularity
Coordinate Singularity i.e.

2D Line element in polar coordinates..

\[ ds^2 = dr^2 + r^2 d\phi^2 \]

Perform a coordinate transformation...

\[ r = \frac{a^2}{r'} \]

2D Line element becomes..

\[ ds^2 = \frac{a^4}{r'^4} (dr'^2 + r'^2 d\phi^2) \]

Notice:

\[ \lim_{r' \to 0} ds^2 \to \infty \]
Einstein’s theory of special relativity

Two Postulates:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in a vacuum is equal to the value $c$, independent of the source.
Einstein’s theory of special relativity

Two Postulates:
1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in a vacuum is equal to the value $c$, independent of the source.

Consequences:
1. Time dilation
2. Length contraction
3. Relativity of simultaneity
Line elements are invariant under coordinate transformations...

The flat line element in Cartesian coordinates

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

perform a coordinate transformation to spherical coordinates

\[ ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \]

Same spacetime ‘length’, only coordinates have changed!
⇒ GR is coordinate independent
Line element in Euclidean space...

\[ ds^2 = dx^2 + dy^2 + dz^2 \]

- \( ds^2 \) is the line element measuring length.
- \( ds^2 \) is invariant under rotations.